Interval Estimation of a P(X₁ < X₂) Model for Variables having General Inverse Exponential Form Distributions with Unknown Parameters

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Abstract: In this article the interval estimation of a $P(X_1 < X_2)$ model is discussed when X_1 and X_2 are non-negative independent random variables, having general inverse exponential form distributions with different unknown parameters. Different interval estimators are derived, by applying different approaches. A simulation study is performed to compare the estimators obtained. The comparison is carried out on basis of average length, average coverage, and tail errors. The results are illustrated, using inverse Weibull distribution as an example of the general inverse exponential form distribution.

Keywords: General Inverse Exponential Form Distribution, Coverage Probability, Bootstrap Confidence Interval, Generalized Pivotal Quantity

1. Introduction

There has been continuous interest in the problem of estimating the stress–strength reliability, $R = P(X_1 < X_2)$, where X_1 and X_2 are independent random variables. The parameter R is referred to as the reliability parameter. This problem arises in the classical stress–strength reliability, where the random strength X_2 of a component exceeds the random stress X_1 to which the component is subjected. If $X_1 \geq X_2$, then the component fails. Kotz et al.[1] have surveyed most all the theoretical and the practical results on the theory and applications of the stress-strength reliability problem up to year 2003. However, after year 2003 several authors have considered the stress-strength reliability problem by different approaches for example, Al-Mutairi et al. [2], Amiri et al. [3], Rezaei et al. [4], among others.

Recently Mokhlis et al. [5] have obtained the point and the interval estimation of $R = P(X_1 < X_2)$ by different methods, under the assumption that X_1 and X_2 are non-negative independent and continuous random variables, having the general inverse exponential form with the cumulative distribution functions (CDFs) and probability density functions (pdfs) given respectively by

$$F_{X_i}(x;\eta_i)=\exp[-\eta_i g(x;c)], \text{and}$$
$$f_{X_i}(x;\eta_i)=-\eta_i g'(x;c)\exp[-\eta_i g(x;c)]; i=1,2. \quad (1)$$

where c is a common known parameter, $\eta_i \in \zeta_i$; $i = 1, 2$, are unknown parameters, ζ_i is the parametric space, $g(x; c)$ is a continuous, monotone decreasing, differentiable function, such that, $g(x; c) \to \infty$ as $x \to 0$ and $g(x; c) \to 0$ as $x \to \infty$. They proved that the reliability function, R, is given by $R = \frac{\eta_2}{\eta_1+\eta_2}$, if and only if, X_1 and X_2 have CDFs as in (1). Also, Mokhlis et al. [6] obtained interval estimators of R, when c is unknown, and $\eta_i(b_i, c)$ is a function of the unknown parameters b_i and $c; i = 1,2$. The reliability, R, takes the form

$$R = \frac{\eta_2(b_2,c)}{\eta_1(b_1,c)+ \eta_2(b_2,c)} \quad (2)$$

The present paper, presents estimation of R, when X_1 and X_2 follow the general inverse exponential form distributions with CDFs and pdfs given respectively by

$$F_{X_i}(x; \eta_i, c_i) = \exp[-\eta_i(b_i, c_i)g(x; c_i)], \text{ and}$$
$$f_{X_i}(x; \eta_i, c_i) = -\eta_i(b_i, c_i)g'(x; c_i)\exp[-\eta_i(b_i, c_i)g(x; c_i)]; i = 1,2.$$
(3)

where $\eta_i = \eta_i(b_i, c_i)$ is a differentiable function in two unknown parameters b_i and c_i, where $c_i \in C_i$, $b_i \in B_i$, and C_i and B_i are the parametric spaces of c_i and b_i, respectively. The function $g(x; c_i)$ is a continuous, monotone decreasing, differentiable function, such that, $g(x; c_i) \to \infty$ as $x \to 0$ and $g(x; c_i) \to 0$ as $x \to \infty$, $g'(x; c_i)$ is the first derivative of $g(x; c_i)$ w.r.t x. Using (3), the reliability is given by

$$R = \int_0^\infty -\eta_2(b_2, c_2)g'(z; c_2)\exp[-\eta_1(b_1, c_1)g(z; c_1) - \eta_2(b_2, c_2)g(z; c_2)] dz$$
(4)

The above integral can be evaluated numerically. If $c_1 = c_2 = c$, then the reliability can be expressed as in (2) as Mokhlis et al. [6].

Different interval estimators are constructed by applying different approaches. (i) An approximate confidence interval for R is constructed; using the maximum likelihood estimator (MLE) of R. (ii) A generalized confidence interval is obtained, using the generalized variable (GV) approach. (iii) Two bootstrap confidence intervals (percentile and t) are also presented. (iv) Two Bayesian credible intervals of R are obtained, using Markov chain Monte Carlo (MCMC) method, with different priors. The different interval estimators obtained are illustrated using inverse Weibull distributions as examples of the underlying distributions. A comparison is performed, by means of simulation among the estimators obtained on the basis of average length, average coverage, and tail errors.

This paper is organized as follows: the approximate confidence interval for R is obtained, in Section 2. In Section 3, using generalized variable approach, the generalized confidence interval of R is derived. The bootstrap intervals are obtained, using percentile and t-bootstrap methods, in Section 4. In Section 5, using MCMC method, two Bayesian credible intervals of R are presented by applying two different sets of priors. In Section 6, taking the inverse Weibull distribution as an example of the underlying distributions, the results obtained are illustrated and a numerical comparison of the interval estimators is performed.

2. Approximate Confidence Interval (ACI) of R

Let $\underline{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2$, be two independent random samples from populations with general inverse exponential form distributions given by (3). The likelihood function is

$$L\left(\underline{x}_1, \underline{x}_2 \middle| \eta_1, \eta_2, c_1, c_2\right) = \exp\left[\sum_{i=1}^{2} n_i \ln \eta_i(b_i, c_i) + \sum_{i=1}^{2}\sum_{j=1}^{n_i} \ln\left(-g'(x_{ij}; c_i)\right) - \sum_{i=1}^{2} \eta_i(b_i, c_i)\sum_{j=1}^{n_i} g(x_{ij}; c_i)\right],$$
(5)

where x_{ij} is the jth observation in the sample $\underline{X}_i$; $j = 1, \ldots, n_i$. For simplicity write $\eta_i = \eta_i(b_i, c_i)$; $i = 1,2$, the log-likelihood function is

$$\ln L = \ln L\left(\underline{x}_1, \underline{x}_2 \middle| \eta_1, \eta_2, c_1, c_2\right) = \sum_{i=1}^{2} n_i \ln \eta_i + \sum_{i=1}^{2}\sum_{j=1}^{n_i} \ln\left(-g'(x_{ij}; c_i)\right) - \sum_{i=1}^{2} \eta_i \sum_{j=1}^{n_i} g(x_{ij}; c_i).$$
(6)

To derive the MLEs, $\hat{\eta}_i$, $\hat{c}_i$, and $\hat{b}_i$ of η_i, c_i, and b_i; $i = 1, 2$, respectively, solve the following system of equations for $\hat{\eta}_i, \hat{c}_i$, and $\hat{b}_i$

$$\left.\frac{\partial \ln L}{\partial b_i}\right|_{b_i=\hat{b}_i, c_i=\hat{c}_i, \eta_i=\hat{\eta}_i} = \left.\left(\frac{n_i}{\eta_i} - \sum_{j=1}^{n_i} g(x_{ij}; c_i)\right)\frac{\partial \eta_i}{\partial b_i}\right|_{b_i=\hat{b}_i, c_i=\hat{c}_i, \eta_i=\hat{\eta}_i} = 0, \text{and}$$
(7)

$$\left.\frac{\partial \ln L}{\partial c_i}\right|_{b_i=\hat{b}_i, c_i=\hat{c}_i, \eta_i=\hat{\eta}_i} = \left.\frac{n_i}{\eta_i}\frac{\partial \eta_i}{\partial c_i} + \sum_{j=1}^{n_i}\frac{\partial}{\partial c_i}\ln\left(-g'(x_{ij}; c_i)\right) - \sum_{j=1}^{n_i} g(x_{ij}; c_i)\frac{\partial \eta_i}{\partial c_i} - \eta_i \sum_{j=1}^{n_i}\frac{\partial}{\partial c_i}g(x_{ij}; c_i)\right|_{b_i=\hat{b}_i, c_i=\hat{c}_i, \eta_i=\hat{\eta}_i} = 0.$$
(8)

As known $\frac{\partial \eta_i}{\partial b_i} \neq 0$, from (7) the MLE, $\hat{\eta}_i$ of η_i; $i = 1, 2$, is expressed as

$$\hat{\eta}_i = \frac{n_i}{\sum_{j=1}^{n_i} g(x_{ij}; \hat{c}_i)}; \ i = 1, 2.$$
(9)

The MLE, $\hat{c}_i$ of c_i; $i = 1, 2$, can be obtained by substituting (9) in (8), and then solving (8) numerically. Finally, the

MLE, $\hat{b}_i$ of b_i; $i = 1, 2$, is obtained by using the relation $\hat{\eta}_i = \eta_i(\hat{b}_i, \hat{c}_i)$. Consequently, the MLE, $\hat{R}$ of R can be obtained by replacing the parameters in (4) with their MLEs. This means

$$\hat{R} = \int_0^\infty -\hat{\eta}_2 g'(z; \hat{c}_2)\exp[-\hat{\eta}_1 g(z; \hat{c}_1) - \hat{\eta}_2 g(z; \hat{c}_2)] dz. \quad (10)$$

Clearly, $\hat{b}_i$ doesn't need to be in $\hat{R}$. The MLE, $\hat{R}$ is asymptotically normal with mean R and variance $\sigma_{\hat{R}}^2 = S^t T^{-1}S$, where T^{-1} is the inverse of the Fisher

information matrix, T, of $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (c_1, c_2, b_1, b_2)$, S^t is the transpose of matrix S, where $S = \left[\dfrac{\partial R}{\partial \theta_i}\right]$, and $T = -E\left[\dfrac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right]$, with

$$\frac{\partial^2 \ln L}{\partial c_i^2} = -\frac{n_i}{\eta_i^2}\left(\frac{\partial \eta_i}{\partial c_i}\right)^2 + \sum_{j=1}^{n_i}\frac{\partial^2}{\partial c_i^2}\ln\left(-g'(x_{ij};c_i)\right) - 2\frac{\partial \eta_i}{\partial c_i}\sum_{j=1}^{n_i}\frac{\partial}{\partial c_i}g(x_{ij};c_i) - \eta_i\sum_{j=1}^{n_i}\frac{\partial^2}{\partial c_i^2}g(x_{ij};c_i),$$

$$\frac{\partial^2 \ln L}{\partial c_i \partial b_i} = \frac{\partial^2 \ln L}{\partial b_i \partial c_i} = -\frac{\partial \eta_i}{\partial b_i}\sum_{j=1}^{n_i}\frac{\partial}{\partial c_i}g(x_{ij};c_i) - \frac{n_i}{\eta_i^2}\frac{\partial \eta_i}{\partial b_i}\frac{\partial \eta_i}{\partial c_i},$$

$$\frac{\partial^2 \ln L}{\partial b_i^2} = -\frac{n_i}{\eta_i^2}\left(\frac{\partial \eta_i}{\partial b_i}\right)^2 ; i = 1, 2,$$

$$\frac{\partial^2 \ln L}{\partial b_i \partial b_j} = \frac{\partial^2 \ln L}{\partial c_i \partial c_j} = \frac{\partial^2 \ln L}{\partial b_i \partial c_j} = \frac{\partial^2 \ln L}{\partial c_i \partial b_j} = 0 ; i, j = 1, 2, i \neq j.$$

Clearly, the explicit expression of $\sigma_{\hat{R}}^2$ depends on $\eta_i, g'(x_{ij}; c_i)$, and $g(x_{ij}; c_i)$; $j = 1, \ldots, n_i, i = 1, 2$. The $(1-\alpha)100\%$ ACI for R is $\left(\hat{R} \pm z_{(1-\alpha/2)}\hat{\sigma}_{\hat{R}}\right)$, where $\hat{\sigma}_{\hat{R}}^2 = S^t T^{-1}S|_{b_i = \hat{b}_i, c_i = \hat{c}_i, \eta_i = \hat{\eta}_i}$ is the estimator of $\sigma_{\hat{R}}^2$.

3. Generalized Confidence Interval (GCI) of R

Let $\underline{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i = 1, 2$, be two independent random samples from populations with distributions given in (3), having unknown parameters c_i and b_i; $i = 1, 2$. It is known that, a GPQ of any parameter is a function of observed statistics and random variables whose distribution is free of unknown parameters. For constructing the GCI, applying a useful feature of the GV approach, which states that if G_{b_i} and G_{c_i} are GPQs of b_i and c_i, then $\eta_i(G_{b_i}, G_{c_i})$ is a GPQ of η_i; $i = 1, 2$. This feature enables us obtaining a GPQ for R given as $G_R = R(G_{\eta_1}, G_{\eta_2}, G_{c_1}, G_{c_2})$ by replacing the parameters in (4) with their GPQs, then using G_R in constructing confidence interval for R. The $(1-\alpha)100\%$ GCI for R is obtained as $\left(G_{R(\alpha/2)}, G_{R(1-\alpha/2)}\right)$, where $G_{R(\alpha/2)}$ and $G_{R(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R.

4. Bootstrap Confidence Interval (boot) of R

There are several ways to construct bootstrap confidence intervals. Clearly, the percentile and t-bootstrap confidence intervals are commonly used for the reliability, (see, Efron [7]). Algorithm 1 is applied for constructing the bootstrap confidence intervals.

Algorithm 1.
1. From the original data $\underline{x}_i$, compute the MLEs, $\hat{c}_i$ and $\hat{\eta}_i$ of c_i and η_i; $i = 1, 2$, respectively, and the MLE, $\hat{R}$ of R.
2. Resample two independent random samples $\underline{x}_i^*$; $i = 1, 2$, from $\underline{x}_i$; $i = 1, 2$, respectively; compute the MLEs, $\hat{c}_1^*, \hat{c}_2^*, \hat{\eta}_1^*, \hat{\eta}_2^*$, and $\hat{R}^*$ of c_1, c_2, η_1, η_2, and R, respectively.
3. Repeat step 2, N times to derive $\{\hat{R}_j^*; j = 1, \ldots, N\}$, and order $\hat{R}_j^*; j = 1, \ldots, N$, like that $\hat{R}_j^{*(1)} \leq \cdots \leq \hat{R}_j^{*(N)}$.
4. Construct the percentile and t-bootstrap confidence intervals of R, as follows:
a. Percentile bootstrap (P-boot)
The $(1-\alpha)100\%$ P-boot for R is given by $\left(\hat{R}_{(\alpha/2)}^*, \hat{R}_{(1-\alpha/2)}^*\right)$, where $\hat{R}_{(\alpha/2)}^*$ and $\hat{R}_{(1-\alpha/2)}^*$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of $\hat{R}^*$, respectively.
b. T-bootstrap (T-boot)
The $(1-\alpha)100\%$ T-boot for R is expressed as $\left(\hat{R} - \hat{t}_{(1-\alpha/2)}S^*, \hat{R} - \hat{t}_{(\alpha/2)}S^*\right)$, where S^* is the sample standard deviation of $\{\hat{R}_j^*; j = 1, \ldots, N\}$ and $\hat{t}_{(\alpha)}$ is the (α)th quantile of $\left\{\dfrac{\hat{R}_j^* - \hat{R}}{S^*}; j = 1, \ldots, N\right\}$.

5. Bayesian Credible Interval (BCI) of R

a. Gamma priors (G-BCI)
Let $\underline{X}_i$; $i = 1, 2$ be two independent random samples from general inverse exponential form distributions in (3) with unknown parameters b_i and c_i. Consider, $\eta_i = \eta_i(b_i, c_i)$, as a single parameter, assume that the prior distributions of η_i; $i = 1, 2$ are independent with pdfs

$$\pi_i(\eta_i) = \frac{\beta_i^{\alpha_i}}{\Gamma\alpha_i}\eta_i^{\alpha_i - 1}e^{-\beta_i\eta_i} ; i = 1, 2, \eta_i, \alpha_i, \beta_i > 0. \qquad (11)$$

Moreover assume that, c_i; $i = 1, 2$ have independent gamma priors with pdfs

$$\pi_{ii}(c_i) = \frac{\lambda_i^{\mu_i}}{\Gamma\mu_i}c_i^{\mu_i - 1}e^{-\lambda_i c_i} ; i = 1, 2, c_i, \mu_i, \lambda_i > 0. \qquad (12)$$

The joint posterior distribution of η_i and c_i; $i = 1, 2$ is as follows

$$\pi\left(\eta_1,\eta_2,c_1,c_2 \mid \underline{x}_1,\underline{x}_2\right) = \frac{L\left(\underline{x}_1,\underline{x}_2\middle|\eta_1,\eta_2,c_1,c_2\right)\pi_1(\eta_1)\pi_2(\eta_2)\pi_{11}(c_1)\pi_{22}(c_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L\left(\underline{x}_1,\underline{x}_2\middle|\eta_1,\eta_2,c_1,c_2\right)\pi_1(\eta_1)\pi_2(\eta_2)\pi_{11}(c_1)\pi_{22}(c_2)d\eta_1 d\eta_2 dc_1 dc_2} \tag{13}$$

It is observed that (13) cannot be obtained in a closed form. The BCI can be computed by a combination of Metropolis-Hastings and Gibbs sampling. Moreover the marginal posterior distribution of η_i is

$$\pi_i\left(\eta_i \mid c_i,\underline{x}_1,\underline{x}_2\right) = \frac{\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)^{(n_i+\alpha_i)}}{\Gamma\left(n_i+\alpha_i\right)}\eta_i^{(n_i+\alpha_i-1)}\exp\left[-\eta_i\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right], \tag{14}$$

which is $\mathrm{Gamma}\left((n_i+\alpha_i),\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right)$; i=1, 2. The marginal posterior distribution of c_i is

$$\pi_{ii}\left(c_i\middle|\underline{x}_1,\underline{x}_2\right) = A_i^{-1}\exp\left[(\mu_i-1)\ln c_i - c_i\lambda_i + \sum_{j=1}^{n_i}\ln\left(-g'(x_{ij};c_i)\right)-\left(n_i+\alpha_i\right)\ln\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right], \tag{15}$$

where $A_i = \int_0^\infty \exp\left[(\mu_i-1)\ln c_i - c_i\lambda_i + \sum_{j=1}^{n_i}\ln\left(-g'(x_{ij};c_i)\right)-\left(n_i+\alpha_i\right)\ln\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right]dc_i; i=1,2.$

Notice that the marginal posterior distribution of c_i; i = 1,2, is not known and so the Metropolis-Hastings with Gibbs sampling algorithm can be used to solve it as follows (see, Asgharzadeh et al. [8]).

Algorithm 2.

1. Choose a starting values $c_i^{(0)}$; i = 1, 2.
2. For j=1 to N times.
3. Generate $\eta_i^{(j)}$ from Gamma $\left((n_i+\alpha_i),\left(\beta_i+\sum_{j=1}^{n_i}g\left(x_{ij};c_i^{(j-1)}\right)\right)\right)$; i=1, 2, respectively.
4. Generate $c_i^{(j)}$ from (15) using the Metropolis-Hastings algorithm with the normal proposal distribution $\rho_i\sim N(c_i^{(j-1)},1)$; i = 1,2.
   a. Generate ξ_i from the proposal distribution ρ_i; i = 1,2.
   b. Define $Q_i = \min\left\{1,\frac{\pi_{ii}(\xi_i|\underline{x}_1,\underline{x}_2)\rho_i(c_i^{(j-1)})}{\pi_{ii}(c_i^{(j-1)}|\underline{x}_1,\underline{x}_2)\rho_i(\xi_i)}\right\}$; i = 1,2.
   c. Generate u_i from Uniform (0, 1). Take
   $$c_i^{(j)} = \begin{cases} \xi_i; & u_i \leq Q_i, \\ c_i^{(j-1)}; & \text{otherwise} \end{cases}$$
5. Calculate the R^j, using
   $$R^j = \int_0^\infty -\eta_2^{(j)}g'\left(z;c_2^{(j)}\right)\exp\left[-\eta_1^{(j)}g\left(z;c_1^{(j)}\right)-\right.$$

$\quad\eta_2^{(j)}g\left(z;c_2^{(j)}\right)\bigg]dz.$

6. End j loop.
7. Order $R^j; j = 1,\dots,N$, in ascending ordered to obtain $R^{j(1)} \leq \cdots \leq R^{j(N)}$.
8. Construct the $(1-\alpha)100\%$ G-BCI for R of gamma priors as $\left(\widetilde{R}_{g(\alpha/2)},\widetilde{R}_{g(1-\alpha/2)}\right)$, where $\widetilde{R}_{g(\alpha/2)}$ and $\widetilde{R}_{g(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R, respectively.

b. Mixed priors (M-BCI)

Assume that $\underline{X}_i, i = 1,2$ are two independent random samples from populations with (3) having unknown parameters b_i and c_i; i = 1,2, and also assume that, η_i; i = 1,2 has independent gamma prior as in (11), and c_i has uniform improper prior distribution with pdf

$$\pi_{ii}(c_i) = 1; i=1,2, c_i > 0.$$

The joint posterior distribution of η_i and c_i; i = 1, 2 cannot be obtained in a closed form. The marginal posterior of η_i is obtained as (14), while the marginal posterior distribution of c_i is given by

$$\pi_{ii}\left(c_i\middle|\underline{x}_1,\underline{x}_2\right) = B_i^{-1}\exp\left[\sum_{j=1}^{n_i}\ln\left(-g'(x_{ij};c_i)\right)-\left(n_i+\alpha_i\right)\ln\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right], \tag{16}$$

where $B_i = \int_0^\infty \exp\left[\sum_{j=1}^{n_i}\ln\left(-g'(x_{ij};c_i)\right)-\left(n_i+\alpha_i\right)\ln\left(\beta_i+\sum_{j=1}^{n_i}g(x_{ij};c_i)\right)\right]dc_i; i=1,2.$

The marginal posterior distribution of c_i; i = 1,2, is not a known form. To obtain the $(1-\alpha)100\%$ M-BCI for R of mixed priors, use Algorithm 2 with a difference in step 4 by generating $c_i^{(j)}$ from (16) instead of (15).

The $(1-\alpha)100\%$ M-BCI for R is $\left(\widetilde{R}_{m(\alpha/2)},\widetilde{R}_{m(1-\alpha/2)}\right)$, where $\widetilde{R}_{m(\alpha/2)}$ and $\widetilde{R}_{m(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R, respectively.

6. Numerical Illustration

This section, presents a numerical illustration of the results obtained. The confidence intervals; ACI, GCI, P-boot, T-boot, G-BCI, and M-BCI for R with some general inverse exponential form distributions are compared. 1000 samples of sample sizes $(n_1, n_2) = (10, 10)$ and $(30, 30)$ from the underlying distributions of X_1 and X_2 , with unknown

parameters are generated. Taking $\alpha = 0.05$, average length, average coverage probability, left and right tail errors of the $(1 - \alpha)100\%$ confidence intervals are calculated. The parameter values that produce the values of R to be approximately 0.6, 0.7, 0.8, 0.9, 0.95, 0.97, and 0.99 are selected.

The inverse Weibull distribution is chosen as an example of the general inverse exponential form. The inverse Weibull distribution is flexible and includes a variety of distributions. For the inverse Weibull distributions, the CDFs with $\eta_i(b_i, c_i) = \frac{1}{b_i^{c_i}}$, $g(x; c_i) = \frac{1}{x^{c_i}}$, and $g'(x; c_i) = \frac{-c_i}{x^{c_i+1}}$; $i = 1,2$, are

$$F_{X_i}(x; b_i, c_i) = \exp\left[-\left(\frac{1}{b_i x_i}\right)^{c_i}\right], \qquad (17)$$

and the reliability is obtained by substituting these values in (4) as

$$R = \int_0^\infty c_2 b_2^{-c_2} z^{-c_2-1} \exp[-b_1^{-c_1} z^{-c_1} - b_2^{-c_2} z^{-c_2}]\, dz. \quad (18)$$

As noticed from (9), $\hat{\eta}_i = \frac{1}{\hat{b}_i^{\hat{c}_i}} = \frac{n_i}{\sum_{j=1}^{n_i} x_{ij}^{-\hat{c}_i}}$; $i = 1, 2$, and

hence $\hat{b}_i = \left(\frac{\sum_{j=1}^{n_i} x_{ij}^{-\hat{c}_i}}{n_i}\right)^{\frac{1}{\hat{c}_i}}$. Using the Newton–Raphson iterative method $\hat{c}_i$ is obtained from (8) after applying the suitable substitutions.

(i) The $(1 - \alpha)100\%$ ACI of R can be obtained as $\left(\hat{R} \pm z_{(1-\alpha/2)}\hat{\sigma}_{\hat{R}}\right)$, where $\hat{R}$ is obtained from (10), and $\hat{\sigma}_{\hat{R}}^2 = S^t T^{-1} S|_{b_i=\hat{b}_i, c_i=\hat{c}_i, \eta_i=\hat{\eta}_i}$ is obtained from the following equations.

$$\frac{\partial^2 \ln L}{\partial c_i^2} = -\frac{n_i}{c_i^2} - \frac{1}{b_i^{c_i}} \sum_{j=1}^{n_i} \frac{1}{x_{ij}^{c_i}}\left(\ln\left(b_i x_{ij}\right)\right)^2,$$

$$\frac{\partial^2 \ln L}{\partial c_i \partial b_i} = \frac{\partial^2 \ln L}{\partial b_i \partial c_i} = -\frac{n_i}{b_i} + \frac{1}{b_i^{c_i+1}} \sum_{j=1}^{n_i} \frac{1}{x_{ij}^{c_i}} - \frac{c_i}{b_i^{c_i+1}} \sum_{j=1}^{n_i} \frac{1}{x_{ij}^{c_i}} \ln\left(b_i x_{ij}\right),$$

$$\frac{\partial^2 \ln L}{\partial b_i^2} = \frac{n_i c_i}{b_i^2} - \frac{c_i(c_i+1)}{b_i^{c_i+2}} \sum_{j=1}^{n_i} \frac{1}{x_{ij}^{c_i}}; i = 1, 2.$$

(ii) As mentioned in Section 3, taking $G_{c_i} = \left(\frac{c_i}{\hat{c}_i}\right)\hat{c}_{0i} = \frac{\hat{c}_{0i}}{\hat{c}_i^{**}}$, and $G_{b_i} = \left(\frac{b_i}{\hat{b}_i}\right)^{\frac{1}{G_{c_i}}} \hat{b}_{0i} = \left(\frac{1}{\hat{b}_i^{**}}\right)^{\frac{1}{G_{c_i}}} \hat{b}_{0i}$; $i = 1, 2$, and $\hat{c}_{0i}$ and $\hat{b}_{0i}$ are the observed values of $\hat{c}_i$ and $\hat{b}_i$. The distributions of $\hat{c}_i^{**} = \left(\frac{\hat{c}_i}{c_i}\right)$ and $\hat{b}_i^{**} = \left(\frac{\hat{b}_i}{b_i}\right)$; $i = 1,2$, do not depend on any unknown parameters, and so they are pivotal quantities (see, Thoman et al. [9]). $\hat{c}_i^{**}$ and $\hat{b}_i^{**}$ are the MLEs of c_i and b_i, respectively, based on two independent random samples from standard inverse exponential distributions (see, Krishnamoorthy et al. [10]). Then consequently the GPQs of η_i are $G_{\eta_i} = \left(\frac{1}{G_{b_i}}\right)^{G_{c_i}}$. The GCI is obtained by computing the $G_R = R(G_{\eta_1}, G_{\eta_2}, G_{c_1}, G_{c_2})$. The next Algorithm 3 is used to estimate the GCI of R (see, Krishnamoorthy and Lin [11]).

Algorithm 3.

1. Let $\underline{x}_i$; $i = 1, 2$, be two independent random samples from (17), respectively. Compute the $\hat{c}_{0i}$ and $\hat{b}_{0i}$ of c_i and b_i; $i = 1, 2$.
2. Generate two independent random samples $\underline{x}_i^{**}$; $i = 1, 2$, from standard inverse exponential distributions. Compute the $\hat{c}_i^{**}$ and $\hat{b}_i^{**}$ of c_i and b_i.
3. Compute the $G_{c_i}, G_{b_i}, G_{\eta_i}$, and hence G_R; $i = 1, 2$.
4. Repeat the steps 2-3, N times to obtain a set of values of G_R, say $\{G_{R_j}; j = 1, \dots, N\}$.
   Order G_{R_j}; $j = 1, \dots, N$, ascending to obtain $G_{R_j}^{(1)} \le \cdots \le G_{R_j}^{(N)}$.
5. Construct the $(1 - \alpha)100\%$ GCI of R as $\left(G_{R(\alpha/2)}, G_{R(1-\alpha/2)}\right)$.

(iii) The $(1 - \alpha)100\%$ P-boot and T-boot of R are obtained, using Algorithm 1.

Table 1. *Average length of the confidence intervals of R for inverse Weibull distributions, $\eta_i = \frac{1}{b_i^{c_i}}$; $i = 1, 2$.*

	R	Average length											
		n=10						n=30					
		ACI	GCI	boot		BCI		ACI	GCI	boot		BCI	
				P	T	Gamma	Mixed			P	T	Gamma	Mixed
$c_1 = 2, c_2 = 5, b_1 = 1.2, b_2 = 0.9$	0.6207	0.4605	0.4512	0.4812	0.4812	0.3909	0.3947	0.2759	0.2915	0.2814	0.2814	0.2517	0.2530
$c_1 = 3, c_2 = 4, b_1 = 1.2, b_2 = 0.9$	0.7050	0.4161	0.4230	0.4180	0.4180	0.3931	0.3875	0.2525	0.2715	0.2535	0.2535	0.2439	0.2408
$c_1 = 2, c_2 = 3, b_1 = 2, b_2 = 1$	0.8068	0.3429	0.3748	0.3042	0.3042	0.3741	0.3591	0.2106	0.2394	0.2066	0.2066	0.2206	0.2131
$c_1 = 2, c_2 = 3, b_1 = 2.8, b_2 = 0.9$	0.9127	0.2223	0.2822	0.1767	0.1767	0.3199	0.2803	0.1341	0.1666	0.1280	0.1280	0.1633	0.1475
$c_1 = 2, c_2 = 3, b_1 = 4, b_2 = 0.9$	0.9557	0.1455	0.2128	0.1118	0.1118	0.2857	0.2366	0.0865	0.1170	0.0828	0.0828	0.1264	0.1105
$c_1 = 2, c_2 = 3, b_1 = 6, b_2 = 0.9$	0.9799	0.0855	0.1511	0.0628	0.0628	0.2489	0.1992	0.0498	0.0745	0.0482	0.0482	0.0919	0.0764
$c_1 = 2, c_2 = 3, b_1 = 9, b_2 = 0.9$	0.9910	0.0513	0.1088	0.0422	0.0422	0.2175	0.1699	0.0274	0.0451	0.0277	0.0277	0.0663	0.0529

Table 2. *Average coverage probability of the confidence intervals of R for inverse Weibull distributions,* $\eta_i = \frac{1}{b_i^{c_i}}; i = 1, 2.$

	R	Average coverage probability											
		n=10						n=30					
		ACI	GCI	boot		BCI		ACI	GCI	boot		BCI	
				P	T	Gamma	Mixed			P	T	Gamma	Mixed
$c_1 = 2, c_2 = 5, b_1 = 1.2, b_2 = 0.9$	0.6207	0.887	0.951	0.884	0.838	0.985	0.973	0.907	0.944	0.904	0.866	0.956	0.945
$c_1 = 3, c_2 = 4, b_1 = 1.2, b_2 = 0.9$	0.7050	0.865	0.942	0.867	0.811	0.954	0.959	0.893	0.941	0.879	0.853	0.924	0.933
$c_1 = 2, c_2 = 3, b_1 = 2, b_2 = 1$	0.8068	0.835	0.94	0.793	0.746	0.874	0.963	0.874	0.944	0.864	0.82	0.914	0.947
$c_1 = 2, c_2 = 3, b_1 = 2.8, b_2 = 0.9$	0.9127	0.794	0.923	0.732	0.666	0.741	0.915	0.871	0.953	0.846	0.801	0.865	0.937
$c_1 = 2, c_2 = 3, b_1 = 4, b_2 = 0.9$	0.9557	0.772	0.919	0.718	0.614	0.627	0.879	0.834	0.942	0.826	0.755	0.787	0.907
$c_1 = 2, c_2 = 3, b_1 = 6, b_2 = 0.9$	0.9799	0.721	0.918	0.688	0.588	0.513	0.809	0.83	0.942	0.835	0.718	0.713	0.861
$c_1 = 2, c_2 = 3, b_1 = 9, b_2 = 0.9$	0.9910	0.695	0.922	0.681	0.55	0.421	0.71	0.789	0.926	0.801	0.684	0.642	0.82

Table 3. *Left tail error of the confidence intervals of R for inverse Weibull distributions,* $\eta_i = \frac{1}{b_i^{c_i}}; i = 1, 2.$

	R	Left tail error											
		n=10						n=30					
		ACI	GCI	boot		BCI		ACI	GCI	boot		BCI	
				P	T	Gamma	Mixed			P	T	Gamma	Mixed
$c_1 = 2, c_2 = 5, b_1 = 1.2, b_2 = 0.9$	0.6207	0.2219	0.2540	0.2435	0.2210	0.2373	0.1956	0.1318	0.1536	0.1270	0.1421	0.1416	0.1222
$c_1 = 3, c_2 = 4, b_1 = 1.2, b_2 = 0.9$	0.7050	0.1961	0.2555	0.2145	0.1796	0.2930	0.2282	0.1238	0.1550	0.1212	0.1274	0.1670	0.1331
$c_1 = 2, c_2 = 3, b_1 = 2, b_2 = 1$	0.8068	0.1658	0.2562	0.1552	0.1376	0.3083	0.2530	0.1035	0.1475	0.0999	0.1031	0.1631	0.1359
$c_1 = 2, c_2 = 3, b_1 = 2.8, b_2 = 0.9$	0.9127	0.1158	0.2240	0.1013	0.0846	0.3016	0.2342	0.0677	0.1152	0.0689	0.0604	0.1384	0.1101
$c_1 = 2, c_2 = 3, b_1 = 4, b_2 = 0.9$	0.9557	0.0783	0.1816	0.0712	0.0517	0.2821	0.2156	0.0459	0.0883	0.0495	0.0384	0.1156	0.0912
$c_1 = 2, c_2 = 3, b_1 = 6, b_2 = 0.9$	0.9799	0.0475	0.1363	0.0440	0.0284	0.2516	0.1916	0.0273	0.0604	0.0322	0.0209	0.0885	0.0685
$c_1 = 2, c_2 = 3, b_1 = 9, b_2 = 0.9$	0.9910	0.0297	0.1019	0.0336	0.0168	0.2224	0.1685	0.0154	0.0384	0.0200	0.0110	0.0656	0.0496

Table 4. *Right tail error of the confidence intervals of R for inverse Weibull distributions,* $\eta_i = \frac{1}{b_i^{c_i}}; i = 1, 2.$

	R	Right tail error											
		n=10						n=30					
		ACI	GCI	boot		BCI		ACI	GCI	boot		BCI	
				P	T	Gamma	Mixed			P	T	Gamma	Mixed
$c_1 = 2, c_2 = 5, b_1 = 1.2, b_2 = 0.9$	0.6207	0.2385	0.1972	0.2376	0.2601	0.1535	0.1991	0.1441	0.1379	0.1544	0.1393	0.1100	0.1307
$c_1 = 3, c_2 = 4, b_1 = 1.2, b_2 = 0.9$	0.7050	0.2200	0.1674	0.2035	0.2383	0.1000	0.1592	0.1286	0.1164	0.1323	0.1261	0.0769	0.1076
$c_1 = 2, c_2 = 3, b_1 = 2, b_2 = 1$	0.8068	0.1771	0.1186	0.1489	0.1665	0.0657	0.1061	0.1071	0.0918	0.1066	0.1035	0.0575	0.0771
$c_1 = 2, c_2 = 3, b_1 = 2.8, b_2 = 0.9$	0.9127	0.1065	0.0582	0.0753	0.0920	0.0183	0.0460	0.0664	0.0513	0.0591	0.0676	0.0248	0.0374
$c_1 = 2, c_2 = 3, b_1 = 4, b_2 = 0.9$	0.9557	0.0672	0.0312	0.0406	0.0601	0.0036	0.0210	0.0406	0.0287	0.0332	0.0443	0.0108	0.0192
$c_1 = 2, c_2 = 3, b_1 = 6, b_2 = 0.9$	0.9799	0.0379	0.0147	0.0188	0.0344	-0.0026	0.0076	0.0224	0.0141	0.0160	0.0273	0.0034	0.0079
$c_1 = 2, c_2 = 3, b_1 = 9, b_2 = 0.9$	0.9910	0.0215	0.0068	0.0086	0.0253	-0.0048	0.0013	0.0120	0.0067	0.0076	0.0166	0.0006	0.0032

(iv) Finally, the $(1 - \alpha)100\%$ G-BCI and $(1 - \alpha)100\%$ M-BCI of R, are obtained using Algorithm 2. The hyper-parameters of priors for G-BCI and M-BCI are chosen on basis of the same mean but different variances as the follow

G-BCI: let $(\alpha_1, \beta_1) = (2, 4), (\alpha_2, \beta_2) = (1, 2), (\mu_1, \lambda_1) = \left(\frac{3}{2}, 3\right)$, and $(\mu_2, \lambda_2) = \left(\frac{1}{2}, 1\right)$.

M-BCI: let $(\alpha_1, \beta_1) = (2, 4)$, and $(\alpha_2, \beta_2) = (1, 2)$.

The comparisons on the basis of average length, average coverage, and left and right tail errors are introduced in Tables (1-4), respectively. From Tables 1-4, the following results are observed:

1. As expected, the average length of all confidence intervals decreases when n and R increase.
2. When R = 0.8068-0.9910, the average length of boot and the left tail error of T-boot are smallest, and the left tail error of all confidence intervals decrease when n and R increase.
3. The average coverage of GCI is approximately around the $(1 - \alpha)100\%$.
4. The average coverage of ACI, boot, and G-BCI decrease when R increases. But the average coverage of ACI and boot increase when n increases.
5. For all the confidence intervals, the right tail error is decreasing when R is increasing. Moreover the right tail error of all confidence intervals except BCI are decreasing when n is increasing.
6. The right tail error of BCI is the smallest.

7. Conclusion

In this article, several approaches for estimating the confidence intervals of Stress-Strength Reliability $P(X_1 < X_2)$ are provided when random variables having general inverse exponential form distributions with different unknown parameters. A simulation study is performed to compare its performance using the inverse Weibull distribution as an example of the general inverse exponential form distribution. The comparison is carried out on basis of average length, average coverage, and tail errors. The results obtained were very close to each other.

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