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Bayesian Prediction Based on Type-I Hybrid Censored Data from a General Class of Distributions

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Abstract: One and two-sample Bayesian prediction intervals based on Type-I hybrid censored for a general class of distribution 1-F(x)=[ah(x)+b]^c are obtained. For the illustration of the developed results, the inverse Weibull distribution with two unknown parameters and the inverted exponential distribution are used as examples. Using the importance sampling technique and Markov Chain Monte Carlo (MCMC) to compute the approximation predictive survival functions. Finally, a real life data set and a generated data set are used to illustrate the results derived here.

Keywords: Bayesian Prediction, Type-I Hybrid Censored, General Class, Markov Chain Monte Carlo, Importance Sampling Technique

1. Introduction

Prediction plays an important role in different areas of applied statistics such as medical sciences and reliability analysis. Bayesian prediction have more attention among other issues of prediction. Discussion of the prediction intervals (one-sample and two-sample prediction) for a future sample is valuable in lifetime studies. Bayesian prediction intervals for future observations have been discussed by several authors, including Howlader [1], Geisser [2], Raqab and Nagaraja [3], Al-Hussaini and Jaheen ([4]; [5]), Abdel-Aty et al. [6], Kundu and Howlader [7], Mohie El-Din et al. ([8]; [9]), Shafay and Balakrishnan [10], Mohie El-Din and Shafay [11] and Shafay et al. [12]. In this article, we use a general class of distribution (see; Khan and Abu-Salih [13], Athar and Islam [14]) to derive general procedure for determining the one- and two-sample Bayesian prediction intervals based on Type-I hybrid censored data. In the rest of this section, we derive the likelihood function and the conditional density functions of $X_{s:n}$ given the Type-I hybrid censored data. In Section 2, we derive the one-sample Bayesian predictive survival function and the one-sample Bayesian predictions bounds for the $s-th(r<s\le n)$ ordered lifetime from Type-I hybrid censored sample. Furthermore, we derive the two-sample Bayesian predictive survival function and the two sample Bayesian predictions bounds for the $s-th$ ordered lifetime from a future independent sample. In Section 3, special cases of this general class such as the inverse Weibull distribution when the two parameters are unknown and the inverted exponential distribution are considered as illustrative examples, wherein we adopt the importance sampling technique to compute the approximation predictive survival function in the one-sample case and the Markov Chain Monte Carlo (MCMC) method to compute the approximation predictive survival function in the two-sample case. Finally, some numerical examples are conducted to illustrate the prediction procedures.

Let the general form of distributions be

$$1-F(x)=\left[ah(x)+b\right]^{c}, \quad \alpha\le x\le\beta, \tag{1}$$

where a, b and c are constants (a; c $\neq$ 0) s.t $F(\alpha)=0, F(\beta)=1$ and $h(x)\equiv h(x;\underline{\theta})$ is a monotonic and differentiable function of x in the interval $[\alpha,\beta]$ and the parameter $\underline{\theta}\in\Theta$ may be a real vector, then

$$f(x)=-cah'(x)\left[ah(x)+b\right]^{c-1}, \tag{2}$$

where $h'(x)=\dfrac{d}{dx}h(x)$. The following table gives some distributions with proper choice of a; b; c and h(x) as examples of the general class.

Table 1. *Some distributions derived from the general class.*

Distributions	$\bar{F}(x)$	a	b	c	h(x)
Exponential	$e^{-\lambda x}$ $x>0, \lambda>0$	1	0	λ	e^{-x}
Generalized Exponential	$1-(1-e^{-(x-\mu)\lambda})^{\alpha}$ $x>\mu, \lambda>0, \alpha>0$	-1	1	1	$(1-e^{-(x-\mu)\lambda})^{\alpha}$
Generalized Inverted Exponential	$(1-e^{-(x-\mu)\lambda})^{\alpha}$ $x>\mu, \lambda>0, \alpha>0$	-1	1	α	$e^{-(x-\mu)\lambda}$
Weibull	$e^{-\theta(x-\mu)^{p}}$ $x\geq\mu$	1	0	θ	$e^{-(x-\mu)^{p}}$
Inverse Weibull	$1-e^{-\theta x^{-p}}$ $0\leq x<\infty$	-1	1	1	$e^{-\theta x^{-p}}$
Pareto	$\alpha^{p}x^{-p}$ $\alpha\leq x<\infty$	α	0	p	x^{-1}
Gumbel	$1-e^{-e^{(x-\mu)\beta}}$ $\beta>0$	-1	1	1	$e^{-e^{(x-\mu)\beta}}$
Burr type XII	$(1+x^{c})^{-k}$ $x>0, c>0, k>0$	1	1	$-k$	x^{c}
Beta of the first kind	$(1-x)^{p}$ $0<x<1$	-1	1	p	x

Let $X_{1:n}<X_{2:n}<\cdots<X_{n:n}$ be the order statistics from a random sample of size n from a distribution function $F(x)$ given in (1) with density function $f(x)$ given in (2).

Let K denote the number of $X_{i:n}$'s that are at most T. Then K is a discrete random variable with support $\{0,1,\cdots,n\}$ and probability density function as

$$P(K=k)=\binom{n}{k}p^{k}q^{n-k},\ k=0,1,\cdots,n, \tag{3}$$

where $p=F(T)$ and $q=1-p=1-F(T)$.

We have one of the two following types of observations:

Case I. $X_{1:n}<X_{2:n}<\cdots<X_{r:n}$ if $X_{r:n}\leq T$ with $r\leq K\leq n$;

Case II: $X_{1:n}<X_{2:n}<\cdots<X_{K:n}$ if $T<X_{r:n}$ with $0\leq K\leq r-1$.

The likelihood function of a Type-I hybrid censored sample is as follows:

Case I.

$$L_1(\theta;X_r)=\frac{n!}{(n-r)!}\prod_{i=1}^{r}-cah'(x_i)\left[ah(x_i)+b\right]^{c-1}\left[ah(x_r)+b\right]^{c(n-r)}, \tag{4}$$

where $X_r=(x_1,\cdots,x_r)$ and $x_1<\cdots<x_r\leq T$.

Case II:

$$L_2(\theta;X_k)=\frac{n!}{(n-k)!}\prod_{i=1}^{k}-cah'(x_i)\left[ah(x_i)+b\right]^{c-1}\left[ah(T)+b\right]^{c(n-k)} \tag{5}$$

where $X_k=(x_1,\cdots,x_k)$ and $x_1<\cdots<x_k\leq T<x_{k+1}$.

When $r<s\leq n$, the conditional density function of $X_{s:n}$, given the Type-I hybrid censored data, is obtained as follows:

Case I.

$$f_1(x_s\mid X_r)=\begin{cases} f_{11}(x_s\mid X_r), & x_r<x_s\leq T,\\ f_{12}(x_s\mid X_r), & x_s>T,\end{cases} \tag{6}$$

where

$$\begin{aligned} f_{11}(x_s\mid X_r)&=\frac{1}{P(r\leq K\leq n)}\sum_{k=s}^{n}f(x_s\mid x_r,K=k)P(K=k)\\ &=\sum_{k=s}^{n}\frac{(k-r)!P(K=k)}{(s-r-1)!(k-s)!\sum_{j=r}^{n}P(K=j)}\ \frac{\left[F(x_s)-F(x_r)\right]^{s-r-1}\left[F(T)-F(x_s)\right]^{k-s}f(x_s)}{\left[F(T)-F(x_r)\right]^{k-r}}\\ &=\sum_{k=s}^{n}\sum_{q=0}^{s-r-1}\sum_{w=0}^{k-s}\frac{A_1\varphi(x_r,T)\left[ah(x_r)+b\right]^{c(s-r-1-q)}}{\left[\left(ah(x_r)+b\right)^{c}-\left(ah(T)+b\right)^{c}\right]^{k-r}}cah'(x_s)\left[ah(x_s)+b\right]^{c(q+k-s-w+1)-1}\left[ah(T)+b\right]^{cw}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} f_{12}(x_s\mid X_r)&=\frac{1}{P(r\leq K\leq n)}\sum_{k=r}^{s-1}f(x_s\mid x_r,K=k)P(K=k)\\ &=\sum_{k=r}^{s-1}\frac{(n-k)!P(K=k)}{(s-k-1)!(n-s)!\sum_{j=r}^{n}P(K=j)}\ \frac{\left[F(x_s)-F(T)\right]^{s-k-1}\left[1-F(x_s)\right]^{n-s}f(x_s)}{\left[1-F(T)\right]^{n-k}}\\ &=\sum_{k=r}^{s-1}\sum_{q=0}^{s-k-1}A_2\,\varphi(x_r,T)\,c\,a\,h'(x_s)\left[ah(x_s)+b\right]^{c(q+n-s+1)-1}\left[ah(T)+b\right]^{c(s-q-n-1)}, \end{aligned} \tag{8}$$

with

$$A_1 = \frac{(-1)^{q+w+1}(k-r)!}{q!w!(s-r-1-q)!(k-s-w)!},$$

$$A_2 = \frac{(-1)^{q+1}(n-k)!}{q!(n-s)!(s-k-1-q)!},$$

and

$$\varphi(x_r,T) = \frac{\sum_{i=0}^{k}\binom{n}{k}\binom{k}{i}(-1)^i\left[a(T)+b\right]^{c(i+n-k)}}{\sum_{j=r}^{n}\sum_{u=0}^{j}\binom{n}{j}\binom{j}{u}(-1)^u\left[ah(T)+b\right]^{c(u+n-j)}}.$$

Case II:

$$f_2(x_s \mid X_k) = \frac{1}{P(0\le k\le r-1)}\sum_{k=0}^{r-1} f(x_s \mid x_k, K=k)P(K=k)$$

$$= \sum_{k=0}^{r-1}\frac{(n-k)!P(K=k)}{(s-k-1)!(n-s)!\sum_{j=0}^{r-1}P(k=j)}$$

$$\frac{\left[F(x_s)-F(T)\right]^{s-k-1}\left[1-F(x_s)\right]^{n-s} f(x_s)}{\left[1-F(T)\right]^{n-k}}$$

$$= \sum_{k=0}^{r-1}\sum_{q=0}^{s-k-1} A_2\, \psi(x_r,T)\, c\, a\, h'(x_s)\left[ah(x_s)+b\right]^{c(q+n-s+1)-1}\left[ah(T)+b\right]^{c(s-q-n-1)}, \quad (9)$$

where

$$\psi(x_r,T) = \frac{\sum_{i=0}^{n}\binom{n}{k}\binom{k}{i}(-1)^i\left[ah(T)+b\right]^{c(i+n-k)}}{\sum_{j=0}^{r-1}\sum_{u=0}^{nj}\binom{n}{j}\binom{j}{u}(-1)^u\left[ah(T)+b\right]^{c(u+n-j)}}.$$

2. Bayesian Analysis

Bayesian approach has received a lot of attention for estimating the parameters of statistical distributions and for predicting samples. It makes use of ones prior knowledge about the parameters and also takes into consideration the data available. If ones prior knowledge about the parameter is available, it is suitable to make use of an informative prior but in a situation where one does not have any prior knowledge about the parameter and cannot obtain vital information from experts to this regard, then a non-informative prior will be a suitable alternative to use, Guure et al. [15].

Let the prior distribution denoted by $\pi(\theta;\delta)$, where $\theta\in\Theta$ is the vector of parameters of the distribution under consideration and δ is the vector of prior parameters. Then the posterior density function of θ, can be written as:

Case I.

$$\pi_1^*(\theta \mid X_r) = C_1^{-1}L_1(\theta; X_r)\pi(\theta;\delta), \quad (10)$$

where $X_r = (x_1,\cdots,x_r)$ and $x_1 < \cdots < x_r \le T$; and $C_1 = \int_\theta L_1(\theta; X_r)\pi(\theta;\delta)d\theta.$

Case II.

$$\pi_2^*(\theta \mid X_k) = C_2^{-1}L_1(\theta; X_k)\pi(\theta;\delta), \quad (11)$$

where $X_k = (x_1,\cdots,x_k)$ and $x_1 < \cdots < x_k \le T < x_{k+1}$, and $C_2 = \int_\theta L_2(\theta; X_k)\pi(\theta;\delta)d\theta.$

2.1. One-Sample Bayesian Prediction Intervals

We simply obtain the predictive survival function of $X_{s:n}$ as follows:

Case I.

$$\overline{F}_1^*(x_s \mid X_r) = \begin{cases}\overline{F}_{11}^*(x_s \mid X_r), & x_r < x_s \le T,\\ \overline{F}_{12}^*(x_s \mid X_r), & x_s > T,\end{cases} \quad (12)$$

where

$$\overline{F}_{11}^*(t \mid X_r) = \int_t^T\int_{\theta\in\Theta} f_{11}(x_s \mid X_r)\pi_1^*(\theta \mid x_r)d\theta dx_s + \int_T^\beta\int_{\theta\in\Theta} f_{12}(x_s \mid X_r)\pi_1^*(\theta \mid x_r)d\theta dx_s, \quad (13)$$

and

$$\overline{F}_{12}^*(t \mid X_r) = \int_t^\beta\int_{\theta\in\Theta} f_{12}(x_s \mid X_r)\pi_1^*(\theta \mid x_r)d\theta dx_s. \quad (14)$$

Case II.

$$\overline{F}_2^*(t \mid X_k) = \int_t^\beta\int_{\theta\in\Theta} f_2(x_s \mid X_r)\pi_2^*(\theta \mid x_k)d\theta dx_s. \quad (15)$$

The Bayesian predictive $100(1-\gamma)\%$ interval for $X_{s:n}, r < s \le n$, can be obtained by solving the following two equations:

$$\overline{F}^*(L_{X_{s:n}} \mid X) = 1-\frac{\gamma}{2} \text{ and } \overline{F}^*(U_{X_{s:n}} \mid X) = \frac{\gamma}{2}, \quad (16)$$

where

$$\overline{F}^*(t \mid X) = \begin{cases}\overline{F}_1^*(t \mid X_r), & CaseI,\\ \overline{F}_2^*(t \mid X_k), & CaseII,\end{cases}$$

and $L_{X_{s:n}}$ and $U_{X_{s:n}}$ denote the lower and upper bounds, respectively.

2.2. Two-Sample Bayesian Prediction Intervals

Let us consider a future sample $\{Y_1, Y_2, \cdots, Y_m\}$ of size m, independent of the informative sample $\{X_1, X_2, \cdots, X_n\}$ and let $Y_{1:m} \le Y_{2:m} \le \cdots \le Y_{m:m}$ be the order statistics of the future sample. Suppose we are interested in the predictive density of the order statistic $Y_{s:m}$ of the future sample, given the informative data set $\{X_1, X_2, \cdots, X_n\}$. The probability density function of the s-th order statistic of the future sample of size m from a continuous distribution with the distribution function $F(x)$ and the probability density function $f(x)$ is given by

$$f_{Y_{s:m}}(y\mid\theta) = \frac{m!}{(s-1)!(m-s)!}[F(y)]^{s-1}[1-F(y)]^{m-s} f(y),$$

where $1 \le s \le m$; (see, Arnold et al. [16]).

From (1) and (2), we simply obtain the probability density function of the s-th order statistic from a general class as follows:

$$f_{Y_{s:m}}(y\mid\theta) = \sum_{l=0}^{s-1} C_l cah'(y)(ah(y)+b)^{c(m+l-s+1)-1},$$

where $C_l = \dfrac{(-1)^{l+1} m!}{l!(m-s)!(s-l-1)!}$, and we simply obtain the Bayesian predictive density function of $Y_{s:m}$ as follows:

Case I.

$$f^*_{1Y_{s:m}}(y\mid X_r) = \sum_{l=0}^{s-1} C_l \int_\theta cah'(y)(ah(y)+b)^{c(m+l-s+1)-1}\pi_1^*(\theta\mid X_r)d\theta, \quad (17)$$

Case II.

$$f^*_{2Y_{s:m}}(y\mid X_k) = \sum_{l=0}^{s-1} C_l \int_\theta cah'(y)(ah(y)+b)^{c(m+l-s+1)-1}\pi_2^*(\theta\mid X_k)d\theta. \quad (18)$$

From (17) and (18), we simply obtain the predictive survival function of $Y_{s:m}$ as follows:

Case I.

$$\overline{F}^*_{1Y_{s:m}}(t\mid X_r) = \int_t^\beta f^*_{1Y_{s:m}}(y\mid X_r)dy$$

$$= \sum_{l=0}^{s-1} \frac{C_l}{(m+l-s+1)} \int_\theta (ah(t)+b)^{c(m+l-s+1)} \pi_1^*(\theta\mid X_r)d\theta. \quad (19)$$

Case II.

$$\overline{F}^*_{2Y_{s:m}}(t\mid X_k) = \int_t^\beta f^*_{2Y_{s:m}}(y\mid X_k)dy$$

$$= \sum_{l=0}^{s-1} \frac{C_l}{(m+l-s+1)} \int_\theta (ah(t)+b)^{c(m+l-s+1)} \pi_2^*(\theta\mid X_k)d\theta. \quad (20)$$

Then, the Bayesian predictive $100(1-\gamma)\%$ interval for $Y_{s:m}, 1 \le s \le m,$ can be obtained by solving the following two equations:

$$\overline{F}^*(L_{Y_{s:m}}\mid X) = 1-\frac{\gamma}{2} \text{ and } \overline{F}^*(U_{Y_{s:m}}\mid X) = \frac{\gamma}{2}, \quad (21)$$

where

$$\overline{F}^*_{Y_{s:m}}(t\mid X) = \begin{cases} \overline{F}^*_{1Y_{s:m}}(t\mid X_r), & CaseI, \\ \overline{F}^*_{2Y_{s:m}}(t\mid X_k), & CaseII, \end{cases}$$

and $L_{Y_{s:m}}$ and $U_{Y_{s:m}}$ denote the lower and upper bounds, respectively.

3. Examples

In this section, we discuss the Bayesian prediction of observations from the inverse Weibull distribution when both parameters are unknown and from the inverted exponential distribution. To our knowledge, no one study these distributions for determining the Bayesian prediction intervals for future lifetimes based on an observed Type-I hybrid censored data.

3.1. Inverse Weibull Distribution

In this subsection, we take a special case from this general class, the inverse Weibull distribution, when $h(x) = e^{-\theta x^{-p}}, a=-1, b=1,$ and $c=1$, we provide the posterior density function depend on the maximum likelihood distribution given in (4) and (5). Here, we assumed that the model parameters θ and p follow the independent gamma prior density of the following forms:

$$g_1^*(\theta\mid u_1, v_1) \propto \theta^{u_1-1}e^{-v_1\theta}, \quad \theta > 0,$$

$$g_2^*(p\mid u_2, v_2) \propto p^{u_2-1}e^{-v_2 p}, \quad p > 0,$$

where $u_1, v_1, u_2,$ and v_2 are the hyper-parameters. Then, the joint posterior density function of θ and p, given the Type-I hybrid censored data, can be written as:

Case I.

$$\pi_1^*(\theta, p\mid x_r) \propto \theta^{r+u_1-1} e^{-\theta\left(\sum_{i=1}^r x_i^{-p}+v_1\right)} p^{r+u_2-1}\left[\prod_{i=1}^r x_i^{-p-1}\right] e^{-v_2 p}\left[1-e^{-\theta x_r^{-p}}\right]^{n-r}, \quad (22)$$

where $X_r = (x_1, \cdots, x_r)$ and $x_1 < \cdots < x_r \le T.$

Therefore, the posterior density function of θ and p given X_r can be written as

$$\pi_1^*(\theta, p\mid x_r) \propto \pi_{1\theta}(\theta\mid p, X_r)\pi_{1p}(p\mid X_r)h_1(\theta, p\mid X_r), \quad (23)$$

where $\pi_{1\theta}(\theta\mid p, X_r)$ is a gamma density function with the shape and scale parameters as $r+u_1$ and $\left(\sum_{i=1}^r x_i^{-p}+v_1\right)$, respectively, $\pi_{1p}(p\mid X_r)$ is a proper density function given by

$$\pi_{1p}(p \mid X_r) \propto p^{r+u_2-1}\,[\prod_{i=1}^{r} x_i^{-p-1}]e^{-v_2 p}[\sum_{i=1}^{r} x_i^{-p}+v_1]^{-r-u_1}, \quad (24)$$

and

$$h_1(\theta, p \mid X_r) = [1-e^{-\theta x_r^{-p}}]^{n-r}. \quad (25)$$

Case II.

$$\pi_2^*(\theta, p \mid X_k) \propto \theta^{k+u_1-1}\, e^{-\theta\left(\sum_{i=1}^{k} x_i^{-p}+v_1\right)} p^{k+u_2-1}\left[\prod_{i=1}^{k} x_i^{-p-1}\right] e^{-v_2 p}\left[1-e^{-\theta T^{-p}}\right]^{n-k} \quad (26)$$

where $X_k = (x_1, \cdots, x_k)$ and $x_1 < \cdots < x_k \le T < x_{k+1}$.

Similarly as above, we can write the posterior density function of θ and p given X_k as

$$\pi_2^*(\theta, p \mid x_k) \propto \pi_{2\theta}(\theta \mid p, X_k)\pi_{2p}(p \mid X_k)h_2(\theta, p \mid X_k), \quad (27)$$

where $\pi_{2\theta}(\theta \mid p, X_k)$ is a gamma density function with the shape and scale parameters as $k+u_1$ and $\left(\sum_{i=1}^{k} x_i^{-p}+v_1\right)$, respectively, $\pi_{2p}(p \mid X_r)$ is proper density function given by

$$\pi_{2p}(p \mid X_k) \propto p^{r+u_2-1}\,[\prod_{i=1}^{k} x_i^{-p-1}]e^{-v_2 p}[\sum_{i=1}^{k} x_i^{-p}+v_1]^{-k-u_1}, \quad (28)$$

and

$$h_2(\theta, p \mid X_k) = [1-e^{-\theta T^{-p}}]^{n-k}. \quad (29)$$

3.1.1. One-Sample Bayesian Prediction

The conditional density function of $X_{s:n}$ given the Type-I hybrid censored data, (from Eqs (7), (8), (9)), can be written as:

Case I.

$$f_{11}(x_s \mid X_r) = \sum_{k=s}^{n} \frac{(k-r)!B_1(T)}{(s-r-1)!(k-s)!}(e^{-\theta x_s^{-p}} - e^{-\theta x_r^{-p}})^{s-r-1} p\theta x_s^{-p-1}$$
$$e^{-\theta x_s^{-p}}(e^{-\theta T^{-p}} - e^{-\theta x_s^{-p}})^{k-s}(e^{-\theta T^{-p}} - e^{-\theta x_r^{-p}})^{r-k}$$
$$= \sum_{k=s}^{n}\sum_{q=0}^{s-r-1}\sum_{w=0}^{k-s} \frac{(-1)^{q+w}(k-r)!B_1(T)(e^{-\theta T^{-p}} - e^{-\theta x_r^{-p}})^{r-k}}{(s-r-q-1)!q!(k-s-w)!w!}$$
$$(e^{-\theta x_r^{-p}})^{q}(e^{-\theta T^{-p}})^{k-s-w} p\theta x_s^{-p-1}(e^{-\theta x_s^{-p}})^{w+s-r-q}, \quad (30)$$

where

$$B_1(T) = \frac{\binom{n}{k}(e^{-\theta T^{-p}})^k(1-e^{-\theta T^{-p}})^{n-k}}{\sum_{j=r}^{n}\binom{n}{j}(e^{-\theta T^{-p}})^j(1-e^{-\theta T^{-p}})^{n-j}}.$$

Similarly, From Eq. (8), we can obtain

$$f_{12}(x_s \mid X_r) = \sum_{k=r}^{s-1} \frac{(n-k)!B_1(T)}{(s-k-1)!(n-s)!}(e^{-\theta x_s^{-p}} - e^{-\theta T^{-p}})^{s-k-1}(1-e^{-\theta x_s^{-p}})^{n-s} p\theta x_s^{-p-1} e^{-\theta x_s^{-p}}(1-e^{-\theta T^{-p}})^{k-n}$$

$$= \sum_{k=r}^{s-1}\sum_{l=0}^{s-k-1}\sum_{z=0}^{n-s} \frac{(-1)^{l+z}(n-k)!B_1(T)(1-e^{-\theta T^{-p}})^{k-n}}{l!(s-k-l-1)!z!(n-s-z)!}(e^{-\theta T^{-p}})^{l} p\theta x_s^{-p-1}(e^{-\theta x_s^{-p}})^{s-k-l+z}. \quad (31)$$

Case II.

The conditional density function of $X_{s:n}$ given the Type-I hybrid censored data, in this case, can be written as:

$$f_2(x_s \mid X_r) = \sum_{k=0}^{r-1} \frac{(n-k)!B_2(T)}{(s-k-1)!(n-s)!}(e^{-\theta x_s^{-p}} - e^{-\theta T^{-p}})^{s-k-1}(1-e^{-\theta x_s^{-p}})^{n-s} p\theta x_s^{-p-1} e^{-\theta x_s^{-p}}(1-e^{-\theta T^{-p}})^{k-n}$$

$$= \sum_{k=0}^{r-1}\sum_{w=0}^{s-k-1}\sum_{u=0}^{n-s} \frac{(-1)^{w+u}(n-k)!B_2(T)(1-e^{-\theta T^{-p}})^{k-n}}{(s-k-w-1)!w!u!(n-s-u)!}(e^{-\theta T^{-p}})^{w} p\theta x_s^{-p-1}(e^{-\theta x_s^{-p}})^{s-k-w+u}, \quad (32)$$

where

$$B_2(T) = \frac{\binom{n}{k}(e^{-\theta T^{-p}})^k(1-e^{-\theta T^{-p}})^{n-k}}{\sum_{j=0}^{r-1}\binom{n}{j}(e^{-\theta T^{-p}})^j(1-e^{-\theta T^{-p}})^{n-j}}.$$

Then, we obtain the predictive survival function of $X_{s:n}$ as follows:

$$\overline{F}_{11}^*(t \mid X_r) = \int_0^\infty\int_0^\infty(\int_t^T f_{11}(x_s;\theta,p)dx_s)\pi_1^*(\theta,p \mid X_r)d\theta dp + \int_0^\infty\int_0^\infty(\int_T^\infty f_{12}(x_s;\theta,p)dx_s)\pi_1^*(\theta,p \mid X_r)d\theta dp$$

$$= \int_0^\infty \int_0^\infty \sum_{k=s}^{n} \sum_{q=0}^{s-r-1} \sum_{w=0}^{k-s} \frac{(-1)^{q+w}(k-r)! B_1(T)}{q! w! (s-r-q-1)! (k-s-w)!} \frac{(e^{-\theta T^{-p}} - e^{-\theta x_r^{-p}})^{r-k}}{w+s-r-q} (e^{-\theta x_r^{-p}})^q (e^{-\theta T^{-p}})^{k-s-w}$$

$$[(e^{-\theta T^{-p}})^{w+s-r-q} - (e^{-\theta t^{-p}})^{w+s-r-q}] \pi_1^*(\theta, p \mid X_r) d\theta dp$$

$$+ \int_0^\infty \int_0^\infty \sum_{k=r}^{s-1} \sum_{l=0}^{s-k-1} \sum_{z=0}^{n-s} \frac{(-1)^{l+z}(n-k)!(s-k-1)! B_1(T)}{l! z! (s-r-1)! (s-k-l-1)!} \frac{(1-e^{-\theta T^{-p}})^{k-n}}{(s-k-l+z)(n-s-z)!} (e^{-\theta T^{-p}})^l$$

$$[1-(e^{-\theta T^{-p}})^{s-k-l+z}] \pi_1^*(\theta, p \mid X_r) d\theta dp. \tag{33}$$

and similarly

$$\overline{F}_{12}^*(t \mid X_r) = \int_t^\infty \int_0^\infty \int_0^\infty f_{12}(x_s \mid X_r) \pi_1^*(\theta, p \mid X_r) d\theta dp dx_s$$

$$= \int_0^\infty \int_0^\infty \sum_{k=r}^{s-1} \sum_{l=0}^{s-k-1} \sum_{z=0}^{n-s} \frac{(-1)^{l+z}(n-k)! B_1(T)}{l!(s-k-l-1)! z!} \frac{(1-e^{-\theta T^{-p}})^{k-n}}{(n-s-z)!(s-k-l+z)} (e^{-\theta T^{-p}})^l$$

$$(1-e^{-\theta t^{-p}})^{s-k-l+z} \pi_1^*(\theta, p \mid x_r) d\theta dp. \tag{34}$$

$$\overline{F}_2^*(t \mid X_k) = \int_t^\infty \int_0^\infty \int_0^\infty f_2(x_s \mid X_k) \pi_2^*(\theta, p \mid X_k) d\theta dp dx_s$$

$$= \int_0^\infty \int_0^\infty \sum_{k=0}^{r-1} \sum_{w=0}^{s-k-1} \sum_{u=0}^{n-s} \frac{(-1)^{w+u}(n-k)! B_2(T)}{(s-k-w-1)! w! u!} \frac{(1-e^{-\theta T^{-p}})^{k-n}}{(n-s-u)!(s-k-w+u)} (e^{-\theta T^{-p}})^w$$

$$(1-e^{-\theta t^{-p}})^{s-k-w+u} \pi_2^*(\theta, p \mid x_k) d\theta dp. \tag{35}$$

It does not seem to be possible to compute the probabilities in Eqs (33), (34), (35) analytically. Hence, we use the importance sampling technique, (see, Geweke [17]; Chen and Shao ([18]; [19])) to construct the Bayesian prediction interval. The details are explained below.

Importance Sampling technique

Firstly, we need to prove that the $\pi_p(p \mid data)$ as given in (24) and (28) has a log-concave density function: From (24), the $\ln \pi_{1p}(p \mid X_r)$ without the additive constant is

$$\ln \pi_p(p \mid X_r) = \ln c + (r+u_2-1)\ln p - v_2 p - (p+1)\sum_{i=1}^{r} \ln x_i - (r+u_1)\ln[\sum_{i=1}^{r} x_i^{-p} + v_1],$$

it is easy to show that $\frac{d^2}{dp^2} \ln \pi_{1p}(p \mid X_r) < 0$, which implies that $\pi_{1p}(p \mid data)$ has a log-concave density function.

Since $\pi_{1p}(p \mid data)$ has a log-concave density, using the idea of Devroye [20], it is possible to generate sample from $\pi_{1p}(p \mid data)$. Moreover, since $\pi_{1\theta}(\theta \mid p, data)$ follows gamma, it is quite simple to generate sample from $\pi_{1\theta}(\theta \mid p, data)$. Now we would like to provide the importance sampling procedure to compute the probabilities in Eqs (33), (34), (35).

Algorithm:

- Step1: Generate p from $\pi_{1p}(p \mid data)$ using the method developed by Devroye [20].
- Step2: Generate θ from $\pi_{1\theta}(\theta \mid p, data)$.
- Step3: Repeat Step 1 and Step 2 and obtain $(p_1;\theta_1), (p_2;\theta_2), \cdots, (p_M;\theta_M)$.
- Step4: The approximate value of $\int_0^\infty \int_0^\infty f(\theta, p) \pi_1^*(\theta, p \mid X_r) d\theta dp$ can be obtained as

$$\int_0^\infty \int_0^\infty f(\theta, p) \pi_1^*(\theta, p \mid X_r) d\theta dp = \frac{\frac{1}{M}\sum_{i=1}^{M} f(\theta_i, p_i) h_1(\theta_i, p_i \mid X_r)}{\frac{1}{M}\sum_{i=1}^{M} h_1(\theta_i, p_i \mid X_r)}.$$

Similary, we can use the above algorithm to compute $\int_0^\infty \int_0^\infty f(\theta, p) \pi_2^*(\theta, p \mid X_k) d\theta dp.$

The Bayesian predictive $100(1-\gamma)\%$ interval for $X_{s:n}; r < s \le n;$ can be obtained by solving the two equations given in (16).

3.1.2. Two-Sample Bayesian Prediction

The predictive survival function of $Y_{s:m}$ in this special case is obtained as follows:

Case I.

$$\overline{F}_{1Y_{s:m}}^*(t \mid X_r) = \int_t^\infty f_{1Y_{s:m}}^*(y \mid X_r) dy$$

$$=\sum_{l=0}^{s-1}\frac{C_l}{(m+l-s+1)}\int_0^\infty\int_0^\infty(1-e^{-\theta t^{-p}})^{m+l-s+1}\pi_1^*(\theta,p\mid X_r)d\theta dp. \quad (36)$$

Case II.

$$\overline{F}^*_{2Y_{s:m}}(t\mid X_k)=\int_t^\infty f^*_{2Y_{s:m}}(y\mid X_k)dy$$

$$=\sum_{l=0}^{s-1}\frac{C_l}{(m+l-s+1)}\int_0^\infty\int_0^\infty(1-e^{-\theta t^{-p}})^{m+l-s+1}\pi_2^*(\theta,p\mid X_k)d\theta dp, \quad (37)$$

where $C_l=\dfrac{(-1)^{l+1}m!}{l!(m-s)!(s-l-1)!}$.

By the same importance sample technique, the Bayesian predictive $100(1-\gamma)\%$ interval for $Y_{s:m}, 1\le s\le m$, can be obtained by solving the two equations in (21).

3.2. Inverted Exponential Distribution

The inverted exponential distribution is a special case from inverse Weibull distribution when the shape parameter is known ($p=1$). we provide the posterior density function depend on the maximum likelihood distribution given in (4) and (5), when the shape parameter is $p=1$. It is assumed that the scale parameter has a gamma prior distribution with the shape and scale parameters as u and v, respectively and it has the probability density function

$$\pi(\theta\mid u,v)\propto\theta^{u-1}e^{-v\theta},\quad \theta>0.$$

The posterior density function of θ, given the Type-I hybrid censored data, can be written as:

Case I.

$$\pi_1^*(\theta\mid x_r)=\frac{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\theta^{r+u-1}e^{-\theta\left(\frac{j}{x_r}+\sum_{i=1}^{r}\frac{1}{x_i}+v\right)}}{\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left(\frac{j}{x_r}+\sum_{i=1}^{r}\frac{1}{x_i}+v\right)^{-(r+u)}\Gamma(r+u)}, \quad (38)$$

where $X_r=(x_1,\cdots,x_r)$ and $x_1<\cdots<x_r\le T$.

Case II.

$$\pi_2^*(\theta\mid x_k)=\frac{\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\theta^{k+u-1}e^{-\theta\left(\frac{j}{T}+\sum_{i=1}^{k}\frac{1}{x_i}+v\right)}}{\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\left(\frac{j}{T}+\sum_{i=1}^{k}\frac{1}{x_i}+v\right)^{-(k+u)}\Gamma(k+u)} \quad (39)$$

where $X_k=(x_1,\cdots,x_k)$ and $x_1<\cdots<x_k\le T<x_{k+1}$.

In this case the predictive survival function of $X_{s:n}$ is the same as above when we put $p=1$.

3.2.1. One-Sample Bayesian Prediction

To compute $\int_0^\infty f(\theta)\pi_1^*(\theta\mid X_r)d\theta$ by using the MCMC technique, we use the following procedure:

- Step 1. Generate θ_1 from $\pi_1^*(\theta\mid X_r)$;
- Step 2. Repeat Step 1 and obtain $\theta_1,\theta_2,\ldots,\theta_N$;
- Step 3. The approximate value of $\int_0^\infty f(\theta)\pi_1^*(\theta\mid X_r)d\theta$ is then obtained as

$$\int_0^\infty f(\theta)\pi_1^*(\theta\mid X_r)d\theta=\frac{\sum_{i=1}^{N}f(\theta_i)}{N}.$$

Similarly, we can use the above algorithm to compute $\int_0^\infty g(\theta)\pi_2^*(\theta\mid X_k)d\theta.$

The Bayesian predictive $100(1-\gamma)\%$ interval for $X_{s:n}; r<s\le n;$ can be obtained by solving the two equations given in (16).

3.2.2. Two-Sample Bayesian Prediction

The predictive survival function of $Y_{s:m}$ in this special case is obtained as follows:

Case I.

$$\overline{F}^*_{1Y_{s:m}}(t\mid X_r)=\int_t^\infty f^*_{1Y_{s:m}}(y\mid X_r)dy$$

$$=\sum_{l=0}^{s-1}\frac{C_l}{(m+l-s+1)}\int_0^\infty(1-e^{-\theta t^{-1}})^{m+l-s+1}\pi_1^*(\theta\mid X_r)d\theta. \quad (40)$$

Case II.

$$\overline{F}^*_{2Y_{s:m}}(t\mid X_k)=\int_t^\infty f^*_{2Y_{s:m}}(y\mid X_k)dy$$

$$=\sum_{l=0}^{s-1}\frac{C_l}{(m+l-s+1)}\int_0^\infty(1-e^{-\theta t^{-1}})^{m+l-s+1}\pi_2^*(\theta\mid X_k)d\theta. \quad (41)$$

Then, the Bayesian predictive interval for $Y_{s:m}, 1\le s\le m$, can be obtained by the same manner.

4. Numerical Results

In this section we consider a real life data set which the inverse Weibull distribution fits it well and another generated data set from the inverted exponential distribution to illustrate the methods proposed in the previous sections.

The real data set is given by Dumonceaux and Antle [21], and it represents the maximum flood levels (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvania over 20 four-year periods (1890-1969) as: 0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379,0.324, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265. Maswadah [22] and Singh et al. [23] checked the suitability of the inverse Weibull distribution to this real data set and concluded that the inverse Weibull distribution fits the data very well.

We consider two different Type-I hybrid censoring schemes:

1. When $n=20, r=15$ and $T=0.43$. In this case, the life-test would have terminated at $T<x_{15:20}$, and we have obtained the folloing data: 0.265, 0.269, 0.297, 0.315, 0.324,

0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418 and 0.423;

2. When $n = 20, r = 15$ and $T = 0.47$. In this case, the life-test would have terminated at $x_{15:20} = 0.449$, and we have obtained the following data: 0.265, 0.269, 0.297, 0.315, 0.324, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423 and 0.449.

By using the procedures presented earlier, we construct 95% one-sample Bayesian prediction intervals for order statistics $X_{s:n}, s = 16,\cdots,20$ as well as 95% two-sample Bayesian prediction intervals for order statistics $Y_{s:m}, s = 1,5,10,20$ from a future sample of size $m = 20$. To explore the sensitivity of the predictors with respect to the hyperparameters (u_1, v_1, u_2, v_2), we have considered the following four hyperparameters: $(1,7,3,1), (1,7,3,2), (1,8,3,1), (1,8,3,2)$. Table 2 presents the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:n}, s = 16,\cdots,20$ for these four choices of the hyperparameters. Similarly, the lower and upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}, s = 1,5,10,20$ for the different choices of the hyperparameters are presented in Table 3.

Now we generate another data set to illustrate the predictions results for the inverted exponential distribution, we follow the steps

1. given the set of prior parameters, generate the parameter θ,
2. using the generated population parameter, generate an inverted exponential random sample of size n,
3. follow the procedures presented in Section 2.2 to construct one-sample and two-sample Bayesian prediction intervals based on Type-I hybrid censored data.

Given the set of prior parameters (let $u = 30, v = 11,$), we generated the parameter θ from prior distribution, $\theta = 2.7$ then generated the inverted exponential random sample of size $n = 30$, the generated sample is listed as the following:

0.45175	0.84893	0.95595	1.33698	1.38886	1.42612	1.89285	2.0541	.18509	2.60002	2.64192	2.8265
2.91384	3.34748	3.39207	3.49797	.41589	4.56534	4.67023	5.94493	8.93186	9.01355	9.69029	10.5754
.0716	11.7103	13.3331	78.4651	135.999	848.432						

The corresponding results for the one-sample and two-sample prediction intervals are represented in Tables 4 and 5, respectively.

5. Concluding Remarks

In this paper, we obtained one and two sample prediction bounds based on Type-I hybrid censored data under the general class of distributions. We introduced two examples, the inverse Weibull distribution with unknown two parameters and the inverted exponential distribution, to illustrate the developed results. Bayesian predictive survival function can not be obtained in closed form and so importance sampling technique and Markov Chain Monte Carlo samples, which are then used to compute the approximate predictive survival function. Finally, some numerical results are presented to illustrate the results and we observe the following remarks:

1. From Tables 2-5, we notice that the lengths of the Bayesian prediction intervals are short when there are a large number of observed values. It is clear that when we use the same value of r but larger value of T, the Bayesian prediction intervals become tighter.
2. It is observed that the prediction intervals tend be wider when s increase. This is a natural, since the prediction of the future order statistic that is far a way from the last observed value has less accuracy than that of other future order statistics.
3. It is evident from Tables 2 and 3 that the lower bounds of Bayesian prediction are relatively insensitive to the specification of the hyperparameters (u_1, v_1, u_2, v_2) while the upper bounds are somewhat sensitive.

Table 2. *95% one-sample Bayesian prediction bounds for* $X_{s:n}, s = 16,\cdots,20$ *from inverse Weibull distribution.*

r = 15 and T = 0.43								
(u_1,v_1,u_2,v_2)	**(1,7,3,1)**		**(1,7,3,2)**		**(1,8,3,1)**		**(1,8,3,2)**	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
16	0.43907	0.63118	0.44275	0.68220	0.44132	0.66464	0.44615	0.79875
17	0.45561	0.78217	0.45847	0.78082	0.45652	0.82273	0.46265	0.85957
18	0.47628	0.96853	0.47105	0.81186	0.47338	0.91154	0.48197	0.97904
19	0.49904	1.19722	0.51241	1.34441	0.49287	1.02787	0.50513	1.17879
20	0.53634	1.80118	0.55875	2.22601	0.51996	1.14195	0.56461	2.81921
r = 15 and T = 0.47								
(u_1,v_1,u_2,v_2)	**(1,7,3,1)**		**(1,7,3,2)**		**(1,8,3,1)**		**(1,8,3,2)**	
S	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
16	0.44205	0.53420	0.44617	0.57326	0.44310	0.56012	0.44717	0.59083
17	0.44273	0.54657	0.45013	0.65100	0.45077	0.65790	0.45082	0.66844
18	0.45028	0.67311	0.45330	0.77419	0.49619	0.74618	0.46506	0.80804
19	0.46014	0.84750	0.47350	1.10977	0.50554	0.99213	0.47699	1.07860
20	0.50359	1.70414	0.51569	1.19332	0.51248	1.04670	0.52939	2.33474

Table 3. 95% *two-sample Bayesian prediction bounds for* $Y_{s:m}, s=1,5,10,15,20; m=20$ *from inverse Weibull distribution.*

r = 15 and T = 0.43								
(u_1,v_1,u_2,v_2)	**(1,7,3,1)**		**(1,7,3,2)**		**(1,8,3,1)**		**(1,8,3,2)**	
s	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$
1	0.20628	0.32424	0.18440	0.31540	0.21846	0.33201	0.18620	0.32965
5	0.27808	0.40067	0.26681	0.39713	0.28045	0.39171	0.26439	0.39187
10	0.32206	0.48557	0.32817	0.50129	0.32436	0.46338	0.32665	0.52512
15	0.37542	0.59968	0.38940	0.71546	0.37992	0.62889	0.38733	0.71203
20	0.51570	2.12839	0.53647	2.34615	0.49280	1.73166	0.53828	2.52157
r = 15 and T = 0.47								
(u_1,v_1,u_2,v_2)	**(1,7,3,1)**		**(1,7,3,2)**		**(1,8,3,1)**		**(1,8,3,2)**	
S	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$
1	0.21639	0.31731	0.20012	0.32035	0.21495	0.32431	0.19757	0.31722
5	0.28039	0.38947	0.26345	0.38254	0.27969	0.38454	0.27278	0.39054
10	0.32728	0.46743	0.32356	0.48846	0.32377	0.46018	0.32371	0.49926
15	0.37368	0.57193	0.38515	0.66518	0.37618	0.59236	0.38739	0.68362
20	0.51229	1.94013	0.53482	2.13999	0.49172	1.61468	0.53880	2.30696

Table 4. 95% *one-sample Bayesian prediction bounds for* $X_{s:n}, s=26,\cdots,30$. *from inverted exponential distribution.*

	r=25 and T= 10.6		**r=25 and T= 11.5**	
S	$L_{X_{s:n}}$	$U_{X_{s:n}}$	$L_{X_{s:n}}$	$U_{X_{s:n}}$
26	11.6371	52.0702	9.3644	20.9758
27	13.3179	80.5060	9.8224	36.2551
28	15.4181	138.939	11.3103	70.4118
29	19.5173	371.245	12.4387	194.071
30	28.6091	3685.73	16.8671	2019.12

Table 5. 95% *two-sample Bayesian prediction bounds for* $Y_{s:m}, s=1,5,10,20,30; m=30$ *from inverted exponential distribution.*

	r=25 and T= 10.6		**r=25 and T= 11.5**	
S	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$	$L_{Y_{s:m}}$	$U_{Y_{s:m}}$
1	0.35180	1.28963	0.34616	1.25820
5	0.84871	2.38517	0.82808	2.25091
10	1.37635	3.97899	1.36568	3.83175
20	3.28013	12.5604	3.22973	11.9778
30	20.5459	3083.74	20.1275	3016.79

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