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Estimate of subject specific index of relative performance in 'K' samples

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Abstract: This paper proposes and develops a statistic here termed the 'relative performance index' or the index of relative performance' by subjects both within and between several sampled populations for preferentially rank-ordering subjects by their relative performance in comparison with other subjects from these populations involved in a test or contest. The proposed index would enable decisions on the preferential selection of subjects both within and between various classifications for management purposes. The proposed method enables the estimation of the median and other tiles of not only each of the sampled populations but also the common median of the several populations as functions of the relative performance indices. The method unlike some other methods used for the analysis of many samples is based mostly on individual subjects rather than on only summary indices or averages. Test statistics also based on subject specific relative performance indices are developed to test desired hypothesis concerning population. The proposed indices being subject specific rather than merely summary averages easily enables one to more clearly and succinctly examine individual subjects relative performance or level of seriousness in a condition in comparison with other subjects from the sampled populations thereby providing subject targeted information to better guide any interventionist actions on a condition of research interest. The method is illustrated with some data and shown to compare favorably with some existing methods.

Keywords: Rank-Order, Subject Specific, Relative Performance Index, Preferential Selection, Management, Combined Population

1. Introduction

In statistical analysis of k sample data a lot of attention has often be paid and devoted to measures of central tendency and measures of dispersion for these data sets, their estimation and hypothesis tests concerning them. If these are the only interest of a researcher then the researcher can use any of the familiar statistical methods such as the extended median test, the one-way analysis of variance test, the Friedmans two-way analysis of variance test by ranks, or other such methods to analyze the data (Gibbons,1993;Oyeka,2009;Oyeka et al,2010).But k sample data sets intrinsically contain much more unexplored information than only a few parameters such types of information are the relative relationships between the observations themselves as well as the relationships within and between sampled populations. For example often assessors, decision makes, judges teachers etc may assess or judge samples drawn from several populations of subjects, objects, entitled or conditions and score them both within and between the samples for preferential selection relative to one another to fill vacant positions or to guide management decisions when opportunities or resources are scare or limited. A medical or health researcher or health management official may have data or information by some demographic classifications on subjects or patients on their state of health, medical test results level of concentration of some contaminations, disease load, inquiry levels and other such conditions and may wish to relatively rank-order these subjects by the severity of their conditions both within and between the various demographic classifications to guide decisions on the distribution and use of amenities when supplies are limited. In business, industry and governmental affairs, one may wish to know how various outfits, producers' suppliers and distributors of goods and services such as banks, transport operators, ministries, parastatals etc compare in performance when juxtaposed against one another to guide any interventionist remedial actions by management or supervising body.

The problem before the decision makers is how using these observations to rationally select the required number of subjects, objects or outfits from within and between the groups or populations of available subjects or options to ensure that competition and monocracy are upheld in the presence of scarcity. Here although any desired hypothesis may be tested, this may not however be as important as the need to find appropriate ways to systematically rank-order the subjects or available options according to their level of need or performance in a given test or situation to facilitate judicious selection to achieve a desired objective. This is because although hypothesis testing is important and useful it may often not be as important and useful as the need to find ways to rank-order subjects or objects relative one another for preferential selection both within and between sampled populations of subjects or observation.

This paper proposes to develop an index of relative performance that may be of use in rank-ordering subjects, objects or entities according to performance on tests, experiments or conditions for preferential selection both within and between the populations of interest. Test statistics will be provided to enable the testing of some hypotheses if desired. The proposed method may be used in data analysis even when necessary assumptions for the application of some existing statistical methods may not be satisfied by the data.

2. The Proposed Method

Let x_{ih} be the performance, observation or score by the hth subject,object or entity randomly drawn from the ith population in a test experiment or condition in time or space $for\ 'h' = 1,2,....,n_i, i = 1,2,...,k.$ The populations of interest should be measurements on at least the ordinal scale but may or may not be (a) continuous;(b)independent;(c)numeric and (d) of equal sizes.

To develop the proposed method we would first pool the 'k' samples into one combined sample of size $n = \sum_{i=1}^{k} n_i$. Now let

$$u_{lh} = \left.\begin{array}{c} 1, if \ \ x_l > x_h \\ 0, f \ \ x_l = x_h \\ -1, f \ \ x_l < x_h \end{array}\right\} \quad (1)$$
$$for \ h, l = 1, 2, \ldots, n; h \neq l$$

Thus u_{lh} assumes the values 1,0 or -1 respectively if the performance or score by the lth subject, object or entity is higher (better, more),the same (equal to) or lower (worse, less) than the performance or score by the 'hth subject drawn from the combined sample, $h, l = 1, 2, ..., n; h \neq l$.

Let

$$\pi_l^+ = P(u_{lh} = 1); \pi_l^0 = P(u_{lh} = 0); \pi_l^- = P(u_{lh} = -1) \quad (2)$$

Where

$$\pi_l^+ + \pi_l^0 + \pi_l^- = 1 \quad (3)$$

Also let

$$W_l = \sum_{\substack{h=1 \\ h \neq l}}^{n} u_{lh} \quad (4)$$

Now

$$E(u_{lh}) = \pi_l^+ - \pi_l^-; Var(u_{lh}) = \pi_l^+ + \pi_l^- - \left(\pi_l^+ - \pi_l^-\right)^2 \quad (5)$$

Also

$$E\left(W_i\right) = \sum_{\substack{h=1 \\ h \neq l}}^{n} E(u_{lh}) = (n-1)(\pi_l^+ - \pi_l^-) \quad (6)$$

And

$$Var\left(W_i\right) = \sum_{\substack{h=1 \\ h \neq l}}^{n} Var(u_{lh}) = (n-1)\left(\pi_l^+ + \pi_l^- - \left(\pi_l^+ - \pi_l^-\right)^2\right) \quad (7)$$

Now $\pi_l^+, \pi_l^0\ and\ \pi_l^-$ are respectively the probabilities that the lth randomly selected subjects from the pooled population performs or scores higher (better, more) the same as (as well as) or lower (worse, less)than all the other subjects in the combined population for $l = 1, 2, ..., n.$ The sample estimates of these probabilities are respectively

$$\hat{\pi}_l^+ = \frac{f_l^+}{n-1}; \hat{\pi}_l^0 = \frac{f_l^0}{n-1}; \hat{\pi}_l^- = \frac{f_l^-}{n-1} \quad (8)$$

Where $f_l^+, f_l^0\ and\ f_l^-$ are respectively the number of 1s,0s and -1s in the frequency distribution of the n-1 values of these numbers in $u_{lh}, h, l = 1, 2, ..., n; h \neq l.$ Thus $f_l^+, f_l^0\ and\ f_l^-$ are respectively the total number of subjects from the combined sampled populations where scores are less equal to or more than the score by the lth subject. Therefore the sample estimate of the total number of subjects whose scores are lower (less) than the total number of subjects whose scores are higher than the scores by the lth subject or the so called 'index' of relative performance by the lth subject in comparison with all other subjects from the combined population is

$$W_l = (n-1)\left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right) = f_l^+ - f_l^- \quad (9)$$

The corresponding sample variance is from Equations 7 and 8

$$Var(W_l) = (n-1)\left(\hat{\pi}_l^+ + \hat{\pi}_l^- - \left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right)^2\right) \quad (10)$$
$$for\ l = 1,2,...,n$$

Rank ordering the values of W_l of Equation 9 from the largest to the smallest or smallest to the largest and assigning the largest value of W_l. The rank 1 (or n),the next largest the rank 2(or n-1 and so on until the smallest value is assigned the rank n(or 1) enables one rank-order subjects drawn from the combined population together for preferential selection from the highest performer, achiever or best say to the lowest performer, achiever or worst, or vise versa on the basis of the ranks of the assigned to the values of their relative performance index $W_i; l = 1,2,...,n$. All tied values of W_i are assigned their mean ranks.

Now note that W_i the so called gap in the relative performance or the relative performance index by the lth subject from all the populations combined whose estimates is given by Equation 9 with rank r_l is the total number of subjects from the combined population the performance or score by the lth subject is higher (better, greater)less the total number of subjects from the combined population the subjects performance or score is lower (worse, smaller)than ,for $l = 1,2,....,n.$ If the lth subject performs better or score higher than all other subjects from the combined population and W_{is} are not tied in values then $W_l = n-1 = f_l^+, f_l^- = 0; \hat{\pi}_l^+ = 1, \hat{\pi}_l^- = 0$ and the rank $r_l = 1(or\ n)$ depending on the system of ranking adopted. In the case the lth subject is considered the best, most (least) preferred in the preferential ranking of all subjects from the combined population in terms of their performance or score in the test of interest. If the lth subject performs or scores higher (better, more)than one-half of all the subjects in the combined population and lower(worse, less) than another or one-half of the subjects in the combined population then $W_l = 0, f_l^+ = f_l^-; \pi_l^+ = \pi_l^-$, if n is odd or the two middle-most values of W_l are 1 and -1 respectively so that their sum is 0 if n is even. In the case r_l is assigned the middle most rank and that is the median rank, the corresponding W_l if n is odd or the average of the two middle-most values which is now also 0 if 'n' is even is the median relative performance index and hence the corresponding value if 'n' is odd or the average value if n is even of x_l is then the estimated median of the combined population, $l = 1,2,...,n.$ If on the other hand the lth subject performs worse or score less than all the other subjects from the combined population then $W_l = -(n-1) = f_l^+ = 0; f_l^- = 0; \pi_l^+ = 0; \pi_l^- = 1$ and the rank $r_l = n(or\ 1)$ so that the lth subject is considered the worst, least (most) preferred in the preferential rank-ordering of all the 'n' subjects from the combined population in terms of performance in a given test or experiment. Thus the larger and positive the value of W_l is the more(less) higher rated and preferred is the lth subject relative to all other subjects from the combined population in terms of performance or condition; the smaller and negative the value of W_l ,the less (more)the rating of the lth subject in comparison with all other subjects from the combined population. As already shown above, if for instance n is odds, there are no ties and the lth subject performs or scores higher (better, more) than as many subjects from the combined population that subjects performance or score is lower (worse, less) than the $W_l = 0, f_l^+ = f_l^-; \pi_l^+ = \pi_l^-$ and the rank r_l assigned to that subject in the combined ranking of the subjects becomes the median rank of the 'n' sample observations or scores from the combined population. In this case the lth subject is considered better than and more preferred to one-half and worse than and less preferred to another one-half of all the subjects from the combined population. Similarly if 'n' is even then the two middle –most values of W_l are 1 and -1 respectively summing to zero so that their mean rank is the median rank and the average of the corresponding values of the observations is the median value of the combined sample. Thus in particular if the rank-ordering of the values of W_l is from the largest to the smallest then the larger and positive. The value of W_l is the higher, better or greater the performance or score by the lth subject is in comparison with all other subjects from the combined population; and hence the more preferable is the lth subject relative to all other subjects in terms of performance; the smaller and negative the value of W_l is the poorer, lower, worse and hence less preferred is the lth subject in performance, relative to other subjects from the combined population. As already noted above, research interest here may not necessarily be in hypothesis testing but more in rank ordering subjects by their relative performance in a context or condition for possible preferential selection and use for management purposes when opportunities or resources are limited or scarce. One may however still wish to test any desired hypothesis. Now as shown above if in the combined population the performance or score by the lth subject is higher (better or greater) than the scores by one-half of all the other subjects but lower (worse or smaller)than the performance or score by the other one-half of the subjects and the x_l values are not tied and hence the values of W_l are also not tied, then if 'n' is odd W_l would be expected to be zero, that is $W_l = 0$,and if 'n' is even the two middle most values of W_l would be expected to be '1' and '-1' respectively summing to zero also and so assigned the median rank. In this case the corresponding value of x_l or the average of the two middle-most values of x_l would be the sample estimate of the median of the combined population. Now a null hypothesis that may be of interest would be that a randomly selected subject from the combined population performs averagely in the given test or that the subject performs better(worse) than one-half and worse (better)than the other one-half of all the subjects from the combined population so that $E(W_l) = (n-1)(\pi_l^+ - \pi_l^-) = 0.$ In which case we would have that $\pi_l^+ - \pi_l^- = 0.$ This hypothesis may however be stated under a more general null hypothesis as

$$H_0 : \pi_l^+ - \pi_l^- \geq \theta_{lo} \ versus \ H_1 : \pi_l^+ - \pi_l^- < \theta_{lo}, say \ (-1 \leq \theta_{lo} \leq 1) \\ l = 1, 2, \ldots, n \tag{11}$$

The null hypothesis of Equation 11 is tested using the test statistic

$$\chi^2 = \frac{\left(W_l - (n-1).\theta_{lo}\right)^2}{(n-1)\left(\hat{\pi}_l^+ + \hat{\pi}_l^- - (\hat{\pi}_l^+ - \hat{\pi}_l^-)^2\right)} = \frac{(n-1)\left((\hat{\pi}_l^+ - \hat{\pi}_l^-) - \theta_{lo}\right)^2}{\hat{\pi}_l^+ + \hat{\pi}_l^- - (\hat{\pi}_l^+ - \hat{\pi}_l^-)^2} \tag{12}$$
$$for \ l = 1,2,\ldots,n$$

Which under H_0 has appropriately the chi-square distribution with 1 degree of freedom for sufficiently large n.The null hypothesis of Equation 11 is rejected at the α level of significance if

$$\chi^2 \geq \chi^2_{1-\alpha;1} \tag{13}$$

Otherwise H_0 is accepted.

It is possible in some circumstances perhaps based on a quota system of preferential selection that the lth subject from the combined population must a priori statistically out-perform or exceed a specified proportion of all the subjects from the combined population, that is have a relative performance index of at least W_{lo} before such a subject can be considered qualified for inclusion (or exclusion) among the subjects preferentially selected from the combined population given the condition of interest. Now if the desired statistical significance level is α then we would have from Equation 12 and 13 that for the lth subject not to qualify for exclusion (or inclusion) the subjects relative performance index $W_l = (n-1)\left(\hat{\pi}_l^+ - \hat{\pi}_l^+\right)$ must be such that

$$\frac{\left(W_l - W_{lo}\right)^2}{(n-1)\left(\hat{\pi}_l^+ + \hat{\pi}_l^- - \left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right)^2\right)} \geq \chi^2_{1-\alpha;1}$$

Hence for the lth subject from the combined population to be qualified for inclusion (or exclusion)the subjects relative performance index W_l or proportion $\hat{\pi}_l^+ - \hat{\pi}_l^-$ must be such that the a priori specified relative performance index W_{lo} for the subject must lie that is be contained or included within the interval

$$W_l \pm \sqrt{(n-1)\left(\left(\hat{\pi}_l^+ + \hat{\pi}_l^-\right) - \left(\hat{\pi}_l^+ + \hat{\pi}_l^-\right)^2\right)} . \chi^2_{1-\alpha;1} \tag{14}$$
$$for = 1,2,\ldots,n$$

Research interest may also be in determining whether any two randomly selected subjects 'l' and 'g' form the combined population; perform equally well that is have equal relative performance indices $W_l \ and \ W_g$ to do this we may let $W_{lg} = W_l - W_g$,whose sample estimate is from Equation 9

$$W_{lg} = W_l - W_g = (n-1)\left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right) - \left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right) \tag{15}$$

The sample estimate of the corresponding variance is equally shown using Equation 10 to be

$$Var(W_{lg}) = Var(W_l - W_g) = Var(W_l) + Var(W_g) \\ = (n-1)\left(\left(\hat{\pi}_l^+ + \hat{\pi}_l^-\right) - \left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right)^2\right) + \left(\left(\hat{\pi}_g^+ + \hat{\pi}_g^-\right) - \left(\hat{\pi}_g^+ - \hat{\pi}_g^-\right)^2\right) \tag{16}$$

Equation 16 is possible because $W_l and \ W_g$ are uncorrelated to show this we have that

$$Cov(W_l;W_g)=E(W_l;W_g)-E(W_l).E(W_g)=E\left(\sum_{\substack{h=1\\h\neq l}}^{n}u_{lh}\right)\left(\sum_{\substack{s=1\\s\neq q}}^{n}u_{gs}\right)-(n-1)^2\left(\pi_l^+-\pi_l^-\right)\left(\pi_g^+-\pi_g^-\right)$$

$$=\sum_{\substack{h=1\\h\neq l}}^{n}\sum_{\substack{s=1\\s\neq q}}^{n}E(u_{lh}.u_{gs})-(n-1)^2\left(\pi_l^+-\pi_l^-\right)\left(\pi_g^+-\pi_g^-\right).$$

Now $u_{lh}.u_{gs}$ can assume only the values 1,0,and -1.It assumes the value 1 if and only if $u_{lh}\ and\ u_{gs}$ both assume the value 1 or the value -1 with probability $\pi_l^+.\pi_g^+ + \pi_l^+.\pi_g^-$; the value 0 if and only if $u_{lh}\ and\ u_{gs}$ both assume the value 0 or u_{lh} assumes the value 0 no matter the value assumed by $u_{lh}\ or\ u_{gs}$ assumes the value 0 no matter the values assumed by u_{lh} with probability $\pi_l^0.\pi_g^0+\pi_l^0(\pi_g^++\pi_g^-)+\pi_g^0(\pi_l^++\pi_l^-)$; and the value -1,if and only if u_{lh} assumes the value 1 and u_{gs} assumes the value -1 or u_{gs} assumes the value 1 and u_{lh} assumes the value -1 with probability $\pi_l^+.\pi_g^-+\pi_l^-.\pi_g^+$; .Hence, collecting terms we have that

$$Cov(W_l;W_g)=(n-1)^2\left((\pi_l^+.\pi_g^++\pi_l^-.\pi_g^-)-(\pi_l^+.\pi_g^-+\pi_l^-.\pi_g^+)\right)-(\pi_l^+.\pi_l^-)+(\pi_g^+-\pi_g^-)=0$$

To test the null hypothesis of interest, that is

$$H_0:\left(\pi_l^+-\pi_l^-\right)-\left(\pi_g^+-\pi_g^-\right)=0\ versus\ H_1:\left(\pi_l^+-\pi_l^-\right)-\left(\pi_g^+-\pi_g^-\right)\neq 0 \tag{17}$$

We may use the test statistic

$$\chi^2=\frac{W_{1g}^2}{Var(W_{lg})}=\frac{(W_l-W_g)^2}{Var(W_l)+Var(W_g)}$$

$$\chi^2=\frac{(n-1)\left((\hat{\pi}_l^+.\hat{\pi}_l^-)+(\hat{\pi}_g^+-\hat{\pi}_g^-)\right)^2}{\left((\hat{\pi}_l^+.\hat{\pi}_l^-)-(\hat{\pi}_l^+-\hat{\pi}_l^-)\right)^2+\left((\hat{\pi}_g^+.\hat{\pi}_g^-)|-(\hat{\pi}_g^+-\hat{\pi}_g^-)\right)^2} \tag{18}$$

$$for\ l,g=1,2,...,n;l\neq g$$

Which under H_0 has approximately the chi-square distribution with 1 degree of freedom for sufficiently large 'n'.The null hypothesis of Equation 17 is rejected at the α level of significance if Equation 13 is satisfied. Otherwise H_0 is accepted.

The researcher may also be interested in estimating the common median of the combined population as a function of, that is using the concept of relative performance index. As already shown above, in the absence of ties; one-half of the subjects in the combined population would be expected to have performed or scored above as below, that is perform better or higher as worse or lower than the other one-half if the relative performance index is zero. This is because if an observed value is the sample median assigned the median rank and hence with a relative performance index value of 0 then one-half of the observations in the sample would be above as below the median. In such a case, the relative performance index of a randomly selected subject drawn from the combined population with a score that corresponds with the median of the combined sample in the absence of ties would then be expected to be zero if 'n' is odd or the two middle–most indices would have values of 1 and -1 respectively averaging to zero if 'n' is even and assigned the median rank. That is this would in effect mean that the relative performance index with the value zero if 'n' is odd or with the two middle most ranked values of 1 and -1 which sum to zero if n is even in the combined sample assigned the median rank would correspond to the median of the combined sample and would be an estimate of the common population median. The expected implication is that the relative performance index of a randomly selected subject in the combined sample with a performance or score of the combined sample would be zero if 'n' is odd or the average of the middle–most indices would be zero if 'n' is even.

Therefore, notationally and specifically to estimate the common median of the combined population assuming there are no tied observations and hence no ties in the values of the relative performance index $W_{l()}$ of the lth subject in the combined sample is ranked $r_l, l=1,2,...,n.$Then we would expect that

$$\left.\begin{array}{c} E\left(W_{l(n+1/2)}\right) = 0 \ if \ n \ is \ odd \\ E\left(W_{l(n/2)}\right) = 1 \\ and \ E\left(W_{l(n/2+1)}\right) = -1 \ or \end{array}\right\} \tag{19}$$

$vise \ versa \ so \ that \ the \ sum \ of \ these \ two \ middle-most$

$indeces \ is \ also \ zero, if \ 'n' \ is \ even, \ for \ l = 1,2,....,n.$

Hence to estimate the common population median in terms of relative performance indices by subjects in the combined sample one-half needs to examine the 'n' series of relative performance indices by subjects in the combined sample to determine the relative performance index that has a value of zero if 'n' is odd or the two indices with values of 1 and -1 respectively summing to 0 if n is even and hence assigned the median rank. The sample value x_l in the combined sample that has been found to have a relative performance index of 0 and hence assigned the median rank if 'n'is odd or the average of the two middle-most values of these observations that have relative performance indices of 1 and -1 respectively and hence also assigned the median rank if 'n' is even would be the median of the combined sample and hence an estimate of the common population median. In general however, especially when there are ties in the data this same procedure is still followed. In this situation the value x_l of an observation in the combined sample corresponding to the relative performance index assigned the median rank if 'n' is odd or the average of the two observations corresponding to the two middle most rank values of the relative performance indices, if 'n' is even, is taken as the median of the combined sample and hence as an estimate of the common population median. It is likely that if the sample size $'n_i', i = 1,2,...,k,$ are two disparate, then the subject specific relative performance index estimated for subjects based on W_l obtained using the combined sample would vary from sample to sample depending on the sample size. Therefore to adjust each subjects estimated relative performance index for this possibility, the index W_l estimated for the subject using the combined sample is multiplied by the ratio of the size $'n_i'$ of the sample to which the subject belongs to the ratio of the common sample size 'n'.

Thus to do this we may define the sample estimate W_{il},the relative performance index of a subject in the ith sample as a function of the subject relative performance index W_l obtained using the combined sample as

$$W_{il} = \left(\frac{n_i}{n}\right).W_l = \left(\frac{n_i}{n}\right).(n-1)(\hat{\pi}_l^+ - \hat{\pi}_l^-) = \left(\frac{n_i}{n}\right).\left(f_l^+ - f_l^-\right) \tag{20}$$

Where sample variance is

$$Var(W_{il}) = \left(\frac{n_i}{n}\right)^2.(n-1)\left(\left(\hat{\pi}_l^+ + \hat{\pi}_l^-\right)\left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right)^2\right) \tag{21}$$

$$for \ l = 1,2,...,n; i = 1,2,...,k.$$

Note that $\sum_{i=1}^{k} W_{il} = \sum_{i=1}^{k}\frac{n_i}{n}W_l = W_l$

Assigning the rank r_{il} to the value W_{il} in the usual way would enable the preferential rank-ordering of subjects both within and between the sample populations according to their adjusted subject specific relative performance indices adjusted to reflect the sampled population's proportionate representation in the common population. Note that it is quite possible that the preferential rank-ordering of subjects based on $W_l \ and \ r_l$ may be different from their rank-ordering based on $W_{il} \ and \ r_{il}$ depending on how disparate the samples sizes are. Now a possible research interest may be to determine whether the lth randomly selected subject from population 'i' performs at least as well as the gth subject randomly drawn from population j, $l,n=1,2,...,n_i; g=1,2,...,n_j; i,j=1,2,...,k; l \neq g$

That is interest may be in the null hypothesis

$$H_0; E(W_{il}) - E(W_{jg}) \geq W_0; W \geq 0 \tag{22}$$

Against any desired alternative hypothesis

To test this null hypothesis we may let

$$W_{ij;\lg} = W_{il} - W_{jg} = \left(\frac{n_i}{n}\right).W_l - \left(\frac{n_j}{n}\right).W_g = \left(\frac{n_i}{n}\right)\left(f_l^+ - f_l^-\right) - \left(\frac{n_j}{n}\right)\left(f_g^+ - f_g^-\right) \quad (23)$$

Between the relative performance indices by the $lth\ and\ gth$ subjects from populations i and j respectively $l = 1,2,...,n_i; g = 1,2,...n_j; i,j = 1,2,...,k.$ The corresponding sample variance is from equation 21 or equivalently using equation 23

$$\begin{aligned} W_{ij;\lg} &= \left(\frac{n_i}{n}\right).W_l - \left(\frac{n_j}{n}\right).W_g \\ &= (n-1)\left(\left(\frac{n_i}{n}\right)\left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right) - \left(\frac{n_j}{n}\right)\left(\hat{\pi}_g^+ - \hat{\pi}_g^-\right)\right) \end{aligned} \quad (24)$$

Be the sample estimate of the difference.

The corresponding sample variance is from Equation 21

$$Var(W_{ij;\lg}) = Var(W_{il} - W_{jg}) = Var(W_{il}) + Var(W_{jg}) = \left(\frac{n_i}{n}\right)^2 Var(W_l) + \left(\frac{n_j}{n}\right)^2 Var(W_g) \quad (25)$$

Or alternatively,

$$\begin{aligned} Var(W_{ij;\lg}) &= \left(\frac{n_i}{n}\right)^2 Var(W_l) + \left(\frac{n_j}{n}\right)^2 Var(W_g) \\ &= (n-1)\left(\left(\frac{n_i}{n}\right)^2 (\hat{\pi}_l^+ + \hat{\pi}_l^- - \left(\hat{\pi}_l^+ - \hat{\pi}_l^-\right)^2\right) + \left(\frac{n_i}{n}\right)^2 \left((\hat{\pi}_g^+ + \hat{\pi}_g^- - \left(\hat{\pi}_g^+ - \hat{\pi}_g^-\right)^2\right) \end{aligned} \quad (26)$$

The null hypothesis of Equation 22 may now be tested using the test statistic

$$\chi^2 = \frac{\left(W_{il} - W_{jg} - W_o\right)^2}{Var(W_{il}) + Var(W_{jg})} = \frac{\left(\left(\frac{n_i}{n}\right)W_l - \left(\frac{n_j}{n}\right)W_g - W_o\right)^2}{\left(\frac{n_i}{n}\right)^2 Var(W_l) + \left(\frac{n_j}{n}\right)^2 Var(W_g)} \quad (27)$$

$$where\ W_{ij} = W_{il} - W_{jg}\ and\ Var(W_{ij}) = \left(\frac{n_i}{n}\right)^2 Var(W_l) + \left(\frac{n_j}{n}\right)^2 Var(W_g)$$

Are given in Equation 25 and 26 respectively for $l = 1,2,....,n_i; g = 1,2,....,n_j; i,j = 1,2,...,k.$

The null hypothesis H_0 of Equation 22 is rejected at the α level of significance if Equation 13 is satisfied, otherwise H_0 is accepted. It is also again possible that perhaps based on a quota system that the lth subject from population 'i'say may be required to statistically exceed at specified α significance level the gth subject from population 'j' before such a subject may be considered qualified for inclusion (exclusion) from the list of preferentially selected subjects given a test or condition. Under this condition therefore the lth subject from population 'i' to be considered for inclusion (or exclusion) in preference to the gth subject from population 'j'. The gap $W_{il} - W_{jg}$ between the two subjects in their relative performance indices much be such that

$$\left(W_{il} - W_{jg}\right) \pm \sqrt{\left(\left(\frac{n_i}{n}\right)^2 .Var\ W_l\right) + \left(\frac{n_j}{n}\right)^2 .Var\ W_g .\chi^2_{1-\alpha;1}} . \quad (28)$$

Includes, that is covers the value W_o

Finally the researcher may wish to determine whether or not the K population has equal population medians. To do this, one only needs to determine the sample estimate of the common population median as described above and following Equation 19,specifically in the absence of ties the x_{il} values of the observations drawn from the ith population whose relative performance indices $W_{il} = \frac{n_i}{n} . W_l$ have a value of zero if 'n' is odd or the average of the two middle-most observations from the population whose two middle-most ranked relative performance indices have values of 1 and -1 respectively, if 'n' even would in fact correspond, that is be the sample estimate of the common population median, for i=1,2,…,k. If there are ties in the data then a similar approach is followed by now examining the middle-most ranked values of the relative performance indices for each sample in their combined ranking to determine those that have values corresponding to that of the combined sample. Such correspondence would enable the conclusion that such samples and hence the populations from which they are drawn have equal medians otherwise the conclusion would be that the population medians are unequal.

As already noted above, research interest here may not necessarily be in hypothesis testing and in determining whether the sampled populations have equal medians. Interest may rather be more in rank-ordering subjects according to subjects specific relative performance indices for preferential selection both with and between the sampled populations to guide management decisions when opportunities or resources are limited or scare. However if determining statistically whether the sampled populations have equal medians, then the researcher can readily apply the Kruskal –Wallis one-way analysis of variance test by ranks in the usual way using the ranks r_l assigned to W_l in the combined ranking of subject relative performance indices as a pooled sample.

But this investigation can be made much more easily and quickly by examining the series of subject specific relative performance indices W_{il} and their associated ranks r_{il} for each sample. If the middle most relative performance indices if 'n' is odd or the average of the two middle most indices if 'n' is even, for all the samples have the same value with that already determined and associated with the common sample median which is an estimate of the common population average of the values. Then the values on the median of the observations corresponding to these indices would be equal and provide estimates of the various population medians and hence an estimate of the common population median. In this case the conclusion would be that the populations have equal median, otherwise it would be concluded that the populations have different medians. Only those samples that satisfy these conditions may be said to have equal population medians.

Also already noted above in practical applications if interest is not necessarily in hypothesis testing but in rank-ordering subjects relative to one another according to their performance or score in an endeavor then the researcher may need to use only untied subjects from each sampled population in the analysis. In this case if a number of subjects are tied in values, then only one value of the observations among each tied set would be used in the analysis so that the effective sample size for this purpose would now be only the total number of untied subjects in each sample. This is because in the ranking and preferential selection of subjects relative to their performance subjects with tied values or observations would have equal relative performance indices and hence treated alike, that is as one set in the preferential selection process.

As already noted above statistical tests for significance sometimes may not be as important and useful as the need to rank-order subjects, objects or entities by their relative performance in tests, experiments or conditions in time or space by sample subjects from within and between populations for possible preferential selection for policy and management purposes when opportunities or resources are limited or scarce. The proposed method would easily enable one achieve such an objective by simply examining for each population the magnitudes and direction of the estimated relative performance indices W_{il} and selecting subjects or sub-sets of subjects from the population with either the highest or the lowest values of these indices depending on ones interest. If however statistical tests for significance of the estimated percentiles and other tiles are of research interest, then one may use any of the test statistics already provided above as appropriate for these purposes.

But to avoid a situation in which the denominators of these Equations are zero because the response or scores by the lth subject from the sampled population is greater (or less) than those of all other subjects, objects or items drawn from the population so that $f_l^+ = n-1 \text{ and } f_l^- = 0 \text{ or } f_l^+ = 0 \text{ and } f_l^- = -(n-1)$ so that $\hat{\pi}_l^+ = 1 \text{ and } \hat{\pi}_l^- = 0$ or vice versa, yielding a meaningless value of the chi-square test statistic, it is recommended that in such cases a correction factor of $\frac{1}{2(n-1)}$ be subtracted from $\hat{\pi}_l^+$ and added to $\hat{\pi}_l^-$ or vice versa depending on which of the two currently has a value of 1(or -1) or value of 0 for that subject before calculating the variance of $W_l, l = 1,2,....,n$.

3. Illustrative Example

We here illustrate the proposed method using the sample data of Table 1 on the lengths of hospitalization of patients in a certain hospital for Malaria, Hypertension and Hepatitis (Table 1).

***Table 1.** Lengths of hospitalization (in days) of Malaria, Hypertension and Hepatitis patients.*

Malaria patients(M)	**No of days**	**Hypertension patients (Hy)**	**No of days**	**Hepatitis patients(He)**	**No of days**
M1	11	Hy1	4	He1	5
M2	3	Hy2	17	He2	3
M3	1	Hy3	5	He3	4
M4	5	Hy4	16	He4	9
M5	2	Hy5	7	He5	10
M6	3	Hy6	9	He6	5
M7	4	Hy7	18	He7	4
M8	2	Hy8	9	He8	6
M9	7	Hy9	17	He9	8
M10	7	Hy10	13	He10	25
M11	3	Hy11	10	He11	7
		Hy12	5	He12	10
		Hy13	4	He13	10
		Hy14	13		

To apply the proposed method we first pool the sample data of Table 1 to obtain a combined sample of size 'n'=38. Equ 1 is now used with these pooled observations to obtain the values of u_l, whose summary values and other statistics are.

***Table 2:** Summary of the values of u_l, (Equation 1) and other statistics*

Patient	**Days**	f_l^+	f_l^0	f_l^-	$\hat{\pi}_l^+$	$\hat{\pi}_l^0$	$\hat{\pi}_l^-$	W_l	r_l	$\left(\frac{n_i}{n}\right)W_l$	r_{il}
M_1	11	30	0	7	0.811	0.000	0.189	23	8	7	9
M_2	3	3	3	3	0.081	0.001	0.838	-28	33	-8	33
M_3	1	0	0	37	0.100	0.108	1.000	-37	38	-11	38
M_4	5	12	4	21	0.324	0.027	0.568	-9	24	-3	25
M_5	2	1	1	35	0.027	0.081	0.946	-34	36.5	-10	36
M_6	3	3	3	31	0.081	0.108	0.835	-28	33.5	-8	33
M_7	4	7	4	26	0.188	0.027	0.703	-19	29	-6	28
M_8	2	1	1	35	0.027	0.081	0.946	34	36.5	-10	36
M_9	7	18	3	16	0.486	0.081	0.432	2	18.5	1	19.5
M_{10}	7	18	3	16	0.486	0.108	0.432	2	18.5	1	19.5
M_{11}	3	3	3	31	0.081	0.000	0.835	-28	33.5	-8	33
H_{y1}	4	7	4	26	0.188	0.108	0.703	-19	29	-7	30
H_{y2}	17	35	0	2	0.946	0.081	0.054	33	3	12	3.5
H_{y3}	5	12	4	21	0.324	0.108	0.568	-9	24	-3	25
H_{y4}	16	33	0	4	0.892	0.081	0.108	29	5	11	5
H_{y5}	7	18	3	16	0.426	0.054	0.432	2	18.5	1	19.5
H_{y6}	9	23	2	12	0.622	0.108	0.324	11	14	4	15
H_{y7}	18	35	0	2	0.973	0.027	0.007	35	2	13	1.5
H_{y8}	9	23	2	12	0.622	0.054	0.324	11	14	4	3.5
H_{y9}	17	34	1	2	0.919	0.027	0.057	32	4	12	6.5
H_{y10}	13	31	1	5	0.838	0.027	0.135	26	6.5	10	9

Patient	Days	f_l^+	f_l^0	f_l^-	$\hat{\pi}_l^+$	$\hat{\pi}_l^0$	$\hat{\pi}_l^-$	W_l	r_l	$\left(\frac{n_i}{n}\right)W_l$	r_{il}
H_{y11}	10	26	3	8	0.703	0.081	0.216	18	10.5	7	25
H_{y12}	5	12	4	21	0.324	0.108	0.568	-9	24	-3	30
H_{y13}	4	7	4	26	0.189	0.108	0.703	-19	29	-7	6.5
H_{y14}	13	31	1	5	0.838	0.027	0.135	26	6.5	10	25
H_{e1}	5	12	4	21	0.324	0.108	0.568	-9	24	-2	36
H_{e2}	3	3	3	31	0.081	0.081	0.835	-28	33.5	-10	30
H_{e3}	4	7	4	26	0.189	0.108	0.703	-19	29	-7	15
H_{e4}	9	23	2	12	0.122	0.054	0.324	11	14	4	12
H_{e5}	10	26	3	8	0.703	0.081	0.216	18	10.5	6	25
H_{e6}	5	12	4	21	0.324	0.108	0.568	-9	24	-3	9
H_{e7}	4	7	4	26	0.189	0.108	0.703	-19	29	7	22
H_{e8}	6	17	0	20	0.459	0.00	0.541	-3	21	-1	18
H_{e9}	8	22	0	15	0.595	0.000	0.405	7	16	2	17
H_{e10}	25	37	0	0	1.000	0.000	0.00	37	1	13	1.5
H_{e11}	7	18	3	16	0.486	0.081	0.432	2	18.5	1	19.5
H_{e12}	10	26	3	8	0.703	0.081	0.216	18	10.5	6	12
H_{e13}	10	26	3	8	0.703	0.081	0.216	18	10.5	6	12

Table 2 summary of the values of $\mu_l(Equ\ 1)$ and other statistics.

Using Equations 8 and 9,we calculate the values of $\hat{\pi}_l^+, \hat{\pi}_l^0, \hat{\pi}_l^-$ and W_l.Other statistics are similarly calculated and the results are shown in Table 2.Interest may be in testing different hypothesis. For example interest may be in determining whether specific malaria patients say have the same relative performance indices within the population of patients. For instance one may wish to know whether patient M_7 with relative performance index of -19 ranked 29 in the combined ranking of all patients and with a length of hospitalization of 4 days differ statistically with malaria patient M_9 or M_{10} with relative performance index of 2 assigned the median rank of 18.5 in the combined ranking of all patients with the common median length of hospitalization of 7 days (Equation 17).To do this we have from Equation 18 that

$$\chi^2 = \frac{\left(2-(19)\right)^2}{37\left((0.189+0.703-(0.189-0.703)^2\right)+\left(0.486+0.432-(0.486-0.432)^2\right)}$$

$$\frac{441}{37\left(0.628+0.915\right)} = \frac{441}{37.091} = 7.725(P\ value = 0.000)$$

Which with 1 degree of freedom is highly statistical significant, showing that malaria patients whose length of hospitalization is consistent with the common median length of 7 days for the combined population of patients have statistically different length of stay and hence may need more care than their malaria patient colleagues hospitalized for only 4 days. Similarly one may wish to compare the relative performance indices of hypertension and hepatitis patients in the combined ranking of the indices for all patients (Equation 22).For instance one may wish to compare the relative performance index of Hypertension patient Hy6 with adjusted relative performance index specific to own sample of patient,4 corresponding to estimated median length of hospitalization of 9 days for this population of patients with the relative performance index of hepatitis patient He_{11} of 1 also specific to own sample with length of hospitalization of 7 days which is also the estimated length of hospitalization for this population of patients. To do this we have from Equation 27 with Wo=0 that

$$\chi^2 = \frac{(4-1)^2}{37\left(\left(\frac{14}{38}\right)^2\left(0.622+0.324-(0.622-0.324)^2\right)+\left(\left(\frac{13}{38}\right)^2\left(0.486+0.432-(0.486-0.432)^2\right)\right)\right)}$$

$$= \frac{9}{37(0.117+0.107)} = \frac{9}{8.288} = 1.086$$

Which with 1 degree of freedom is not statistically significant, indicating that hypertension and hepatitis patients may have equal length of hospitalization.

To estimate the common median of the combined population using the proposed method we notice from Table 2 with "o"=38 which is even that the two middle most ranked subject specific relative performance indices have values of 2 and ranked 19.5 each and correspond with seven (7) days of hospitalization of malaria patients, which therefore is the sample estimate of the common population median, that is the average length of stay of the three populations of patients. However, it is seen from column 11 of Table 2 that for malaria patients, with n=11 which is odd, the middle-most relative performance index is tied at -8,showing that the median length of hospitalization for the sample of malaria patients and hence an estimate of this populations median is 3 days corresponding to the tied subject relative performance indices. For hypertension patients with n=14 the middle-most relative performance indices are 14,4 and 7 with length of hospitalization 9,9 and givin an estimated median length of hospitalization of 9.5 days for this population of patients/similarly for Hepatitis patient with n=13 which is odd, the middle-most ranked subject relative performance index has a value of has a value of '1' and corresponds with a Hepatitis patient whose length of hospitalization is 7 days which in this case a sample estimate of the median length of hospitalization of the population of Hepatitis patients. Hence, it can be concluded on the basis of these findings that the three population of patients have unequal lengths of hospitalization. Note that 3,9.5 and 7 days are respectively the median length of hospitalization that are also determined in the usual way for the malaria, hypertension and hepatitis samples of patients. To determine or confirm whether the sampled populations have equal population medians we here apply the Kruskal-Wallis one-way analysis of variance test by ranks, using the ranks r_l assigned to the sample observations in their combined ranking. The sums of the ranks assigned to malaria, hypertension and hepatitis patients in their combined ranking are respectively $R_{.1} = 309.5, R_{.2} = 190.0, R_{.3} = 241.5$ .Hence with n_1=1,n_2=14 and n_3=13 we have that the Kruskal-Wallis chi-square test statistic is

$$\chi^2 = 12\sum_{j=1}^{3}\frac{R_{.j}^2/n_j}{n(n+1)} - 3(n+1)$$

$$= \frac{12\left((309.5)^2/11 + (190.0)^2/14 + (241.5)^2/13\right)}{38(39)} - 3(39)$$

$$= 12\frac{(15773.104)}{1482} - 117 = 127.717 - 117 = 10.717(P\ value\ 0.000)$$

Which with 2 degree of freedom is statistically significant, indicating our earlier finding using the proposed method that the three populations of patients have different lengths of hospitalization.

4. Summary and Conclusion

We have in this paper tried to develop a statistical measure here termed an 'index of subject relative performance' or 'subject relative performance index' that enables the preferential selection of subjects both within and between the sampled populations on the basis of the subjects' relative performance or score in comparison with other subjects. The rank-ordered index enables the estimation of by how much a randomly selected subjects performance or score is higher (better, more),the same as (equal to)or lower (worse, less)than the performance scores by all other subjects in a test or contest both within and between populations. Statistical methods of estimating the medians and other titles of the sampled populations and of determining the equality or otherwise of these medians have been presented. Some test statistics have also been developed for testing the statistical significance and differences between the relative performance indices. The proposed indices being subject specific's rather than merely summary averages easily enables one more clearly and succinctly examine individual subjects relative performance or level of seriousness in a condition in comparison with other subjects from the sampled populations therefore providing subject targeted information to better guide any interventionist action in a condition of research interest.

The methods are illustrated with some data and the result obtained using the proposed methods are shown to compare favorably with what would have been obtained if the Kruskal-Walli one-way analysis one-way analysis of variance test by ranks had been to analyze the data.

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