

Soliton Distribution in the Ball and Box Cellular Automation Model

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Abstract: Solitons are very important waves in Mathematical Physics. They help model many physical situations for instance tsunami waves, application in optical communication systems, application in plasma physics and application in laser guided technology among others. Despite enormous applications, solving the non-linear equations which have soliton solutions is challenging because the nonlinear characteristic of the system abruptly changes due to some slight changes of valid parameters including time. A remedy to this is to use cellular automation models to model soliton dynamics. Cellular automation models do not change in response to slight changes on parameters. In this paper, using computer simulation, an investigation of the distribution of the number of solitons from the Ball and Box cellular automation model was examined. The distribution of the number of solitons from the two and three colour Ball and Box cellular automation was established. Using the online integer sequence, for the two colour Ball and Box cellular automation model, it was found that the distribution of the number of the solitons is indexed by the binomial coefficients. On the Other hand, for the three colour Ball and Box cellular automation model, it was found that only solitons of lengths one to four possesses the distribution while the other soliton lengths do not possess any distribution function.

Keywords: Solitons, Ball and Box Model, Cellular Automation, Integer Sequence, Binomial Coefficients

1. Introduction

The Ball and Box Soliton (BBS) is cellular automaton model which models soliton movement proposed by Takahashi and Satsuma (1990) [1, 2, 3]. Its state is defined by using an infinite array of boxes and a finite number of balls, with some boxes filled by balls bounded on the left side, but unbounded on the right side. Each box contains at most one ball. The time evolution of the value of the j th cell at time t , and state

$$u_j^t \in (0, 1)$$

where it is assumed that $u_j^t = 0$ for j far enough to the left and to the right is given by;

$$u_j^{t+1} = \begin{cases} 1 & \text{if } u_j^t = 0 \text{ and } \sum_{i=-\infty}^{j-1} u_i^t > \sum_{i=-\infty}^{j-1} u_i^{t+1} \\ 0 & \text{Otherwise} \end{cases}$$

This corresponds to taking a line of boxes, either containing a ball or empty and updating it. In other words, the rule corresponds to taking the leftmost ball out of its box and putting it in the first empty box to its right, then taking the new leftmost ball as long as it has not already been moved at this time step and moving it to the first empty box to its right. We continue with this process until each ball has been moved, this represents one-time step. The following example details the time evolution under the BBS;

We now provide another example of the two-color state. We start with the arbitrary state 0 1 2 2 0 0 0 2 1 1.

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t = 0 0 1 2 2 0 0 0 0 2 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
t = 1 0 0 0 0 1 2 2 0 0 2 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
t = 2 0 0 0 0 0 0 0 0 1 2 0 2 2 0 1 1 0 0 0 0 0 0 0 0 0 0 0
t = 3 0 0 0 0 0 0 0 0 0 1 0 0 2 2 2 1 1 0 0 0 0 0 0 0 0 0 0
t = 4 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 2 2 1 1 2 0 0 0 0 0 0 0
t = 5 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 2 2 0 1 1 2 0 0 0 0
t = 6 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 2 2 0 0 1 1 2 0
t = 7 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 2 2 0 0 0 1 1 2

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Figure 4. Time evolution in mixed state two color box and ball system.

From this example, it was observed that two unstable balls of lengths three unattained stability in the interval $0 < t < 7$. Using two unstable balls, we had formed three stable balls of lengths 1, 2 and 3 respectively. Using Theorem 1. 1, we conclude that the three stable balls formed are solitons.

From the results of examples above, it is easy to see that different initial conditions of the soliton configurations produces different number of solitons. This calls for investigation of the distribution of the number of the solitons given a particular soliton configuration. we examine the distribution of number of solitons when we involve all possible initial soliton configuration in the two color BBS.

2.1. Simulation of Two-Color Ball and Box

On the 2 color BBS we simulated all the possible soliton configuration from the two color BBS model on the given n empty boxes. We started with two empty boxes then produce all possible ways of coloring the boxes with two colors. Each of these possible ways represent one initial configuration. Using the produced initial configurations, we carry out time update on each of the configurations until stable length of balls are obtained. We then count the number of stable balls. We repeat the process for boxes more than two.

Here, a sequence of length n shall represent coloring n boxes with n colors where the colors are the elements of the sequence, for instance the sequence 1212 shall mean color box one with color one, box two with color two, box three with color one and box four with color two.

As an example, for the two boxes with two colors, we have four ways of producing initial soliton configuration, 11, 22, 21, 12. In the configuration each state represents a unique soliton arrangement which we can start with.

2.2. Results and Discussion

Trivially for the one box we have two ways to color it with two colors either 1 or 2. Both of these lead to 1 soliton. This

means that the total number of 1 soliton is 2. Continuing the process to the other number of boxes and using the on-line integer sequence [10], we have the following result;

Conjecture 2.1 Let n be the number of boxes and k be the number of solitons produced in the two color BBS. The two are related by the binomial coefficients;

$$a(n, k) = \begin{cases} \binom{n+1}{2k-1} & \text{if } n+1 \geq 2k-1 \\ 0 & \text{else} \end{cases}$$

provided that $n, k \in \mathbb{Z}^+$.

3. Three Colour Ball and Box System

We now extend the model introduced in previous section by introducing more species of the ball. We suppose that the box can hold at most one ball and there are three kinds of balls which are indexed by integers one to three.

The time evolution rule corresponds to taking the leftmost ball of index one out of its box and putting it in the first empty box to its right, then taking the new leftmost ball of index one as long as it has not already been moved at this time step and moving it to the first empty box to its right. We continue with this process until each ball of index one has been moved, then we repeat this process to balls of index two and index three. This represents one-time step.

3.1. Simulation of Three Colour Ball and Box Model

As in previous section, we start with two empty boxes then produce all possible ways of coloring the two boxes with three colors. This represent initial soliton configuration. We then evolve all initial soliton configurations to count how many solitons of each kind results. We repeat this process for boxes more than two.

Let 1 denote colour number one, 2, colour number two and 3 represents colour number three. Then 12 denotes box 1 colored with colour one and box two colored with colour two. For the two boxes we have 9 ways of producing initial soliton configuration. We continue with this until all states are filled up. The results of this section are given in Section 3.2 below.

3.2. Results and Discussion

Using the on-line integer sequence and data for the three colour BBS, the Conjecture 3.1 below gives the distribution of the number of solitons.

Conjecture 3.1 Let n be the number of boxes and k be the number of solitons produced in the three color BBS. The two are related by the binomial coefficients.

$$a(n, k) = \begin{cases} \binom{n+2}{2} & \text{if } k = 1 \\ 3\binom{n+3}{5} - 2\binom{n+2}{5} & \text{if } k = 2 \\ \binom{n+1}{4} + 10\binom{n+1}{5} + 15\binom{n+1}{5} + 7\binom{n+1}{7} + \binom{n+1}{8} & \text{if } k = 3 \end{cases}$$

No pattern elsewhere detected which implies there is no

specific sequence which can fit the data.

4. Conclusion

In this paper we have successfully investigated the soliton distribution in the two- and three-color Ball and Box model. The soliton distribution on the two-color Ball and Box automation model is index by binomial coefficients while for the three-color model there is distribution only for at most three solitons while for more than three solitons there is no distribution function. An advantage to use the Ball and Box automation model is that we prevent dealing with non-linear systems which challenging to model solitons.

Since the BBS is composed of a finite number of boxes and balls, it can only take on a finite number of patterns. Hence its trajectory is always simple. In the case where the box capacity is one everywhere and only one kind of ball exists, the formula used to calculate the fundamental cycle is explicitly obtained [10, 11] Hence, using the results in this paper, we may get the formula to calculate number of solitons for multi-color solitons for two colors. In the future we wish to address problem of finding distribution functions for multi-color solitons with colors more than 3

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