



Ramadan Group (RG) Transform Coupled with Projected Differential Transform for Solving Nonlinear Partial Differential Equations

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Abstract: In this article a combination of integral transform method (Ramadan group transform) and projected differential transform is considered to solve partial differential equations. The method can easily be applied to many nonlinear problems and is capable of reducing the size of computational work. The fact that the suggested hybrid method solves such nonlinear partial differential equations without using He's polynomials or Adomian's polynomials is a clear advantage over these decomposition methods. Numerical examples are performed by this hybrid method are presented. The results reveal that the suggested method is simple and effective.

Keywords: Integral Transform Method, Projected Differential Transform Method, He Polynomials, Adomian Polynomials, Partial Differential Equations

1. Introduction

An integral transform is a particular kind of mathematical operator. In mathematics, an integral transform is any transform T of the following form

$$T(f(u)) = \int_{t_1}^{t_2} K(t, u) f(t) dt$$

The input of this transform is a function f , and the output is another function Tf . There are numerous useful integral transforms; each is specified by a choice of the function K of two variables.

2. Integral Transforms

Some of these useful integral transforms are the following ones

2.1. The Laplace Transform

In mathematics the Laplace transform is an integral transform named after its discoverer Pierre-Simon Laplace. It takes a function of a positive real variable t (often time) to a function of a complex variable s (frequency).

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

where $f(t)$ is the original function, $F(s)$ is the transformed function and S is a complex number and $t \geq 0$ is a frequency parameter

The Laplace transform converts integral and differential equations into algebraic equations and it is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

2.2. Sumudu Integral Transform

In the early 90's, Watugala in [1] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform, is defined over the set of functions.

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

by the following formula

$$\tilde{f}(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t} dt, \quad u \in (-\tau_1, \tau_2).$$

The constant M must be finite, while τ_1 and τ_2 need not simultaneously exist. The variable u instead of being used as a power to the exponential as in the case of the Laplace transform, is used to factor the variable t in the argument of the function f

Some of the properties were established in [2, 3]. In [4], further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in [5]. In fact it was shown that there is a strong relationship between Sumudu and other integral transforms; see [6].

In particular the relation between Sumudu transform and Laplace transforms was proved in [7]. Further, in [8], the Sumudu transform was extended to the distributions and some of their properties were also studied in [9]. Recently, this transform is applied to solve the system of differential equations; see [10]. Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n ; see [11]. Thus if

$$f(t) = \sum_{n=0}^\infty a_n t^n \text{ then } F(u) = \sum_{n=0}^\infty n! a_n u^n, \text{ see [6].}$$

2.3. Ramadan Group Integral Transform (RGIT) [12]

A new integral RG transform defined for functions of exponential order, is proclaimed. The proposed new integral transform is a generalization of both Laplace and sumudu transforms. We consider functions in the set A , defined by:

$$A = \left\{ f(t) : \exists M, t_1, t_2 > 0 \text{ s.t. } |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\},$$

The RG transform is defined by

$$K(s, u) = RG(f(t)) = \begin{cases} \int_0^\infty e^{-st} f(ut) dt, & 0 \leq u < t_2, \\ 0 & \\ \int_0^\infty e^{-st} f(ut) dt, & t_1 < u \leq 0, \end{cases}$$

This transform which is a generalization of Laplace and Sumudu transforms is introduced by M. A. Ramadan et al. [12] and, accidentally and unpredictably, it was also introduced by Z. H. Khan and W. A. Khan [13] under the name of N-Transform.

Relations between Laplace, Sumudu and RG transforms
Consider

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t) dt \text{ and}$$

$$G(u) = L(f(t)) = \int_0^\infty e^{-t} f(ut) dt$$

are the Laplace and Sumudu integral transforms respectively, then we can write the following theorem

THEOREM 2.1 [12]

$$K(s, I) = F(s),$$

$$K(I, u) = G(u),$$

$$K(s, u) = \frac{1}{u} F\left(\frac{s}{u}\right)$$

THEOREM 2.2 [12]

Let $f(t) \in A$ with RG transform $K(u, s)$. Then

$$RG\left(t \frac{df(t)}{dt}\right) = -u \frac{\partial K(u, s)}{\partial s},$$

$$RG\left(t^2 \frac{d^2 f(t)}{dt^2}\right) = -u^2 \frac{\partial^2 K(u, s)}{\partial s \partial u}.$$

THEOREM 2.3 [12]

Let $f(t) \in A$ with RG transform $K(u, s)$. Then

$$RG(e^{at} f(t)) = K(s - au, u).$$

THEOREM 2.4 [12]

Let $f(t) = t^{x-1} \in A$ with RG transform $K(u, s)$. Then

$$RG(t^{x-1}) = K(s, u) = \frac{u^{x-1}}{s^x} \Gamma(x).$$

Table 1. Laplace, Sumudu and RG transforms of some functions.

$f(t)$	$F(s) = L\{f(t)\}$	$G(u) = S\{f(t)\}$	$RG(f(t)) = K(s, u)$
1	$\frac{1}{s}$	1	$\frac{1}{s}$
t	$\frac{1}{s^2}$	u	$\frac{u}{s^2}$
$\frac{t^{n-1}}{(n-1)!}, n=1, 2, \dots$	$\frac{1}{s^n}$	u^{n-1}	$\frac{u^{n-1}}{s^n}$
$\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0$	$\frac{1}{s^\alpha}$	$u^{\alpha-1}$	$\frac{u^{\alpha-1}}{s^\alpha}$

The Laplace transforms of function derivatives:

$$L[f'(t)] = s L[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

⋮

$$L[f^{(n)}(t)] = s^n L[f(t)] - \sum_{k=1}^n s^{k-1} f^{(n-k)}(0)$$

The sumudu transforms of function derivatives:

$$G_1(u) = S[f'(t)] = \frac{F(u)}{u} - \frac{f(0)}{u}$$

$$G_2(u) = S[f''(t)] = \frac{F(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u}$$

⋮

$$G_n(u) = S[f^{(n)}(t)] = \frac{F(u)}{u^n} - \frac{f(0)}{u^n} - \dots - \frac{f^{(n-1)}(0)}{u}$$

$$= \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}$$

The Ramadan Group of function derivatives:

$$RG\left[\frac{df(t)}{dt}\right] = \frac{sRG[f(t)] - f(0)}{u}$$

$$RG\left[\frac{d^2 f(t)}{dt^2}\right] = \frac{s^2 RG[f(t)] - sf(0) - uf'(0)}{u^2}$$

⋮

$$RG\left[\frac{d^n f(t)}{dt^n}\right] = \frac{s^n RG[f(t)] - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)}{u^{n-k}}$$

3. Differential Transforms

In this section we first illustrate why differential transforms are important in computing solutions for nonlinear partial differential equations. Laplace transform, Sumudu transform or RG transform is by itself, totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. To illustrate these difficulties, we consider a general nonlinear non-homogeneous PDE with initial conditions of the form

$$Du(x, t) + Ru(x, t) + Nu(x, t) = g(x, t),$$

with the initial conditions of the form

$$u(x, 0) = h(x), \quad u_t(x, 0) = f(x),$$

where D is the second order linear differential operator

$D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D , N represents the general non-linear differential operator and $g(x, t)$ is the source term. Taking, for example, the Laplace transform

$$L[Du(x, t)] + L[Ru(x, t)] + L[Nu(x, t)] = L[g(x, t)].$$

Using the differentiation property of the Laplace transform, we have

$$L[u(x, t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^2} L[Ru(x, t)] + \frac{1}{s^2} L[g(x, t)] - \frac{1}{s^2} L[Nu(x, t)].$$

Operating with the Laplace inverse on both sides

$$u(x, t) = G(x, t) - L^{-1}\left[\frac{1}{s^2} L[Ru(x, t) + Nu(x, t)]\right],$$

The nonlinear term $Nu(x, t)$ is computed by

3.1. He's Polynomials

$$Nu(x, t) = \sum_{n=0}^{\infty} p^n H_n(u),$$

for some He's polynomials H_n that are given by

$$H_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i\right) \right]_{p=0}, \quad n = 0, 1, 2, 3, \dots$$

This method is the coupling of the Laplace transform and the homotopy perturbation method (HPM) using He's polynomials and is called HPTM.

Remark:

The rate of convergence of HPTM is faster than HPM.

The nonlinear term $Nu(x, t)$ may also computed by

3.2. Adomian Decomposition Method (ADM)

The nonlinear operator $N(u)$ is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_n),$$

where A_n is an appropriate Adomian's polynomial which can be calculated for all forms of nonlinearity according to specific algorithms formula:

$$A_n(u_0, u_1, u_2, \dots, u_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

Example

$$N(u) = u^2.$$

We using the form

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

$$N(u) = \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3} + \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + \dots$$

This gives Adomian's polynomials for $N(u) = u^2$ by

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ A_5 &= 2u_0u_5 + 2u_1u_4 + 2u_2u_3, \\ &\vdots \end{aligned}$$

This method is known by integral transform decomposition method (Laplace decomposition method LDM, Sumudu decomposition method SDM, ect.). As we can see computing the nonlinear term is somewhat costly. Recently, Sumudu decomposition method and homotopy perturbation methods are presented to solve nonlinear partial differential equations, see for example, M. A. Ramadan and M. S. Al-luhaibi in [14, 15, 16].

4. Projected Differential Transform Method

The basic definitions and operations of projected differential transform method which can be found in many recent works see for example [17, 18] are introduced as follows:

Definition 4.1

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then

$$U(x, k) = \frac{1}{k!} \left[\frac{\partial^k u(x, t)}{\partial t^k} \right]_{t=0} \quad (1)$$

is the transformed function of $u(x,t)$

Definition 4.2

The projected differential inverse transform of $U(x, k)$ is defined as follows:

$$u(x, t) = \sum_{k=0}^{\infty} U(x, k) t^k \quad (2)$$

Then combining equation (1) and (2) we write

$$u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=0} t^k \quad (3)$$

Table 2. Basic operations of PDTM.

Functional Form	Transformed Form
$u(x, t)$	$U(x, k) = \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=0}$
$u(x, t) \pm v(x, t)$	$U(x, k) \pm V(x, k)$
$cu(x, t)$	$cU(x, k)$
$\left[\frac{\partial u(x, t)}{\partial x} \right]$	$\left[\frac{\partial}{\partial x} U(x, k) \right]$
$\left[\frac{\partial^r}{\partial x^r} u(x, t) \right]$	$\frac{(k+r)!}{k!} U(x, k+r)$
$u(x, t)v(x, t)$	$\sum_{r=0}^k U(x, r)V(x, k-r)$

5. Applications to the RGTM Coupled with PDTM

In order to show the effectiveness of the RGTM coupled with PDTM for solving the nonlinear partial differential equations, several examples are demonstrated. For all illustrative examples, we consider the projected differential transform with respect to the variable t . We choose examples that have exact solutions.

Example 5.1

Consider the simple first order partial differential equation

$$\frac{\partial y}{\partial x} = 2 \frac{\partial y}{\partial t} + y, \quad y(x, 0) = 6e^{-3x} \quad (4)$$

Taking Ramadan Group transformation of equation (4) and making use of the initial condition,

$$RG \left[\frac{\partial y}{\partial x} \right] = 2RG \left[\frac{\partial y}{\partial t} \right] + RG[y]$$

That is, $2RG \left[\frac{\partial y}{\partial t} \right] = RG \left[\frac{\partial y}{\partial x} \right] - RG[y]$

Hence, $\frac{2[sRG[y(x, t)] - y(x, 0)]}{u} = RG \left[\frac{\partial y}{\partial x} - y \right]$

Or, $2sRG[y(x, t)] = 2y(x, 0) + uRG \left[\frac{\partial y}{\partial x} - y \right]$

From the initial condition we get,

$$RG[y(x,t)] = \frac{6}{s}e^{-3x} + \frac{u}{2s}RG\left[\frac{\partial y}{\partial x} - y\right]$$

Taking the inverse Ramadan Group transform, to obtain

$$y(x,t) = 6RG^{-1}\left[\frac{1}{s}\right]e^{-3x} + RG^{-1}\left\{\frac{u}{2s}RG\left[\frac{\partial y}{\partial x} - y\right]\right\}$$

$$= 6e^{-3x} + RG^{-1}\left\{\frac{u}{2s}RG\left[\frac{\partial y}{\partial x} - y\right]\right\}$$

Now, we apply the projected differential transform method Then the recurrence relation is:

$$y(x,0) = 6e^{-3x}, \quad y(x, m + 1) = RG^{-1}\left\{\frac{u}{2s}RG[A_m - B_m]\right\} \quad (5)$$

Where $A_m = \frac{\partial y(x, m)}{\partial x}$, $B_m = y(x, m)$ are the projected differential transform of

$$\frac{\partial y(x, t)}{\partial x}, \quad y(x, t)$$

From equation (5) we find that:

$$y(x,0) = 6e^{-3x}$$

$$A_0 = \frac{\partial y(x,0)}{\partial x} = -18e^{-3x}, \quad B_0 = y(x,0) = 6e^{-3x}$$

$$y(x,1) = RG^{-1}\left\{\frac{u}{2s}RG[-24e^{-3x}]\right\} = -12te^{-3x}$$

$$A_1 = 36te^{-3x}, \quad B_1 = 12te^{-3x}, \quad y(x,2) = 12t^2e^{-3x}$$

And so on. Then the solution in the series form is given by

$$y(x,t) = 6e^{-3x} - 12te^{-3x} + 12t^2e^{-3x} + \dots$$

Example 5. 2

Consider the following second order nonlinear partial differential equation

$$\frac{\partial y}{\partial t} = \left(\frac{\partial y}{\partial x}\right)^2 + y \frac{\partial^2 y}{\partial x^2}, \quad y(x,0) = x^2 \quad (6)$$

The exact solution $u(x, t) = \frac{x^2}{1-6t}$

Taking Ramadan Group transform of equation (6) and making use of the initial condition.

$$RG\left[\frac{\partial y}{\partial t}\right] = RG\left[\left(\frac{\partial y}{\partial x}\right)^2\right] + RG\left[y \frac{\partial^2 y}{\partial x^2}\right]$$

$$\frac{sRG[y(x,t)] - y(x,0)}{u} = RG\left[\left(\frac{\partial y}{\partial x}\right)^2 + y \frac{\partial^2 y}{\partial x^2}\right]$$

$$sRG[y(x,t)] = y(x,0) + uRG\left[\left(\frac{\partial y}{\partial x}\right)^2 + y \frac{\partial^2 y}{\partial x^2}\right]$$

$$RG[y(x,t)] = \frac{1}{s}[x^2] + \frac{u}{s}RG\left[\left(\frac{\partial y}{\partial x}\right)^2 + y \frac{\partial^2 y}{\partial x^2}\right]$$

Applying the inverse Ramadan Group transform implies that

$$y(x,t) = x^2 + RG^{-1}\left\{\frac{u}{s}RG\left[\left(\frac{\partial y}{\partial x}\right)^2 + y \frac{\partial^2 y}{\partial x^2}\right]\right\}$$

Using the projected differential transform method, this leads to the recursive relation

$$y(x,0) = x^2$$

$$y(x, m + 1) = RG^{-1}\left\{\frac{u}{s}RG[A_m + B_m]\right\}$$

where

$$A_m = \sum_{r=0}^m \frac{\partial y(x,r)}{\partial x} \frac{\partial y(x, m-r)}{\partial x},$$

$$B_m = \sum_{r=0}^m y(x,r) \frac{\partial^2 y(x, m-r)}{\partial x^2}$$

are the projected differential transforms of

$$\left(\frac{\partial y}{\partial x}\right)^2 \quad \text{and} \quad y \frac{\partial^2 y}{\partial x^2}$$

we find that:

$$A_0 = \left(\frac{\partial y(x,0)}{\partial x}\right)\left(\frac{\partial y(x,0)}{\partial x}\right) = 4x^2,$$

$$B_0 = y(x,0)\left(\frac{\partial^2 y(x,0)}{\partial x^2}\right) = 2x^2$$

Hence,

$$y(x,1) = RG^{-1}\left\{\frac{u}{s}\right\}(6x^2) = 6tx^2$$

$$A_1 = 48tx^2, \quad B_1 = 24tx^2$$

$$y(x,2) = RG^{-1}\left\{\frac{u}{s}RG[t](72x^2)\right\} = 36t^2x^2$$

And so on, then the solution of equation (6) is

$$y(x,t) = y(x,0) + y(x,1) + y(x,2) + \dots$$

$$y(x,t) = x^2 + 6x^2t + 36x^2t^2 + \dots = x^2(1-6t)^{-1} = \frac{x^2}{1-6t}$$

We can see that the GRTM coupled with PRDTM gives also the exact solution for example 5.2.

Example 5.3

Consider the nonlinear Boussinesq equation

$$U_t + V_x + UU_x = 0, \tag{7}$$

$$V_t + (VU)_x + U_{xxx} = 0, \tag{8}$$

$$U(x, 0) = 2x, V(x, 0) = x^2,$$

with the exact solution

$$U(x, t) = \frac{2x}{1 + 3t}, \quad V(x, t) = \frac{x^2}{(1 + 3t)}$$

Apply RGT on equations (7) and (8) respectively, we get

$$\frac{S}{u} RG[U(x, t)] - \frac{V(x, 0)}{u} = -RG(V_x) - RG(UU_x),$$

$$V(x, t) = x^2 - (RG)^{-1} \left[\frac{u}{s} RG(VU_x) \right] - (RG)^{-1} \left[\frac{u}{s} (V_x U) \right] - (RG)^{-1} \left[\frac{u}{s} RG(U_{xxx}) \right],$$

Using the projected differential transform method, this leads to the recursive relations

$$U(x, 0) = 2x,$$

$$U(x, m + 1) = -(RG)^{-1} \left[\frac{u}{s} RG(A_m) \right] - (RG)^{-1} \left[\frac{u}{s} RG(B_m) \right],$$

where

$$A_m = \frac{\partial V(x, m)}{\partial x}, \quad B_m = \sum_{r=0}^m U(x, r) \frac{\partial U(x, m-r)}{\partial x}, \quad m = 0, 1, \dots$$

and

$$V(x, 0) = x^2,$$

$$V(x, m + 1) = -(RG)^{-1} \left[\frac{u}{s} RG(C_m) \right] - (RG)^{-1} \left[\frac{u}{s} RG(D_m) \right] - (RG)^{-1} \left[\frac{u}{s} RG(E_m) \right],$$

where

$$C_m = \sum_{r=0}^m V(x, r) \frac{\partial U(x, m-r)}{\partial x}, \quad D_m = \sum_{r=0}^m \frac{\partial V(x, m-r)}{\partial x} U(x, r), \quad E_m = \frac{\partial^3 U(x, m)}{\partial x^3}.$$

Now, for $m = 0$, we get

$$A_0 = \frac{\partial V(x, 0)}{\partial x} = 2x, \quad B_0 = U(x, 0) \frac{\partial U(x, 0)}{\partial x} = 4x,$$

Thus,

$$\frac{S}{u} RG[V(x, t)] - \frac{V(x, 0)}{u} = -RG(VU_x) - RG(V_x U) - RG(U_{xxx}),$$

Using the initial conditions, the above two equations can be reduced to

$$RG[U(x, t)] = \frac{2x}{s} - \frac{u}{s} RG(V_x) - \frac{s}{u} RG(UU_x),$$

$$RG[V(x, t)] = \frac{x^2}{s} - \frac{u}{s} RG(VU_x) - \frac{u}{s} RG(V_x U) - \frac{u}{s} RG(U_{xxx}),$$

Now, apply the inverse Ramadan group transform leads to

$$U(x, t) = 2x - (RG)^{-1} \left[\frac{u}{s} RG(V_x) \right] - (RG)^{-1} \left[\frac{u}{s} RG(UU_x) \right],$$

$$\begin{aligned}
 U(x, 1) &= -(RG)^{-1} \left[\frac{u}{s} RG(2x) \right] - (RG)^{-1} \left[\frac{u}{s} RG(4x) \right] \\
 &= -(RG)^{-1} \left[\frac{2ux}{s^2} \right] - (RG)^{-1} \left[\frac{4ux^2}{s^2} \right] \\
 &= -2x (RG)^{-1} \left[\frac{u}{s^2} \right] - 4x (RG)^{-1} \left[\frac{u}{s^2} \right] \\
 &= -2xt - 4xt = -6xt.
 \end{aligned}$$

Also, we have

$$V(x, 1) = -(RG)^{-1} \left[\frac{u}{s} RG(C_0) \right] - (RG)^{-1} \left[\frac{u}{s} RG(D_0) \right] - (RG)^{-1} \left[\frac{u}{s} RG(E_0) \right],$$

where,

$$\begin{aligned}
 C_0 &= V(x, 0) \left[\frac{\partial U(x, 0)}{\partial x} \right] = 2x^2, \\
 D_0 &= \frac{\partial U(x, 0)}{\partial x} U(x, 0) = 4x^2, \quad E_0 = \frac{\partial^3 U(x, 0)}{\partial x^3} = 0,
 \end{aligned}$$

thus,

$$\begin{aligned}
 V(x, 1) &= -2x^2 (RG)^{-1} \left[\frac{u}{s^2} \right] - 4x^2 (RG)^{-1} \left[\frac{u}{s^2} \right], \\
 &= -2x^2 t - 4x^2 t = -6x^2 t.
 \end{aligned}$$

For $m = 1$, we have

$$\begin{aligned}
 A_1 &= \frac{\partial V(x, 1)}{\partial x} = -12xt, \\
 B_1 &= U(x, 0) \frac{\partial U(x, 1)}{\partial x} + U(x, 1) \frac{\partial U(x, 0)}{\partial x} \\
 &= (2x)(-6t) + (-6xt)(2) = -24xt
 \end{aligned}$$

Hence,

$$\begin{aligned}
 U(x, 2) &= -(RG)^{-1} \left[\frac{u}{s} RG(A_1) \right] - (RG)^{-1} \left[\frac{u}{s} RG(B_1) \right], \\
 &= -(RG)^{-1} \left[\frac{u}{s} RG(-12xt) \right] - (RG)^{-1} \left[\frac{u}{s} RG(-24xt) \right],
 \end{aligned}$$

$$\begin{aligned}
 &= -(RG)^{-1} \left[-12x \frac{u^2}{s^3} \right] - (RG)^{-1} \left[24x \frac{u^2}{s^3} \right], \\
 &= 12x \frac{t^2}{2} + 24x \frac{t^2}{2} = 18xt.
 \end{aligned}$$

Similarly, we can get $U(x, 3), U(x, 4), \dots$,

Finally, the solution $U(x, t)$ takes the form

$$\begin{aligned}
 U(x, t) &= U(x, 0) + U(x, 1) + U(x, 2) + \dots \\
 U(x, t) &= 2x - 6xt + 18xt^2 - 54xt^3 + \dots + (-1)^n 3^n 2xt^n + \dots \\
 &= \sum_{n=0}^{\infty} (-1)^n 3^n 2xt^n = \frac{2x}{1+3t}. \tag{9}
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 C_1 &= V(x, 0) \frac{\partial U(x, 1)}{\partial x} + V(x, 1) \frac{\partial U(x, 0)}{\partial x} \\
 &= (x^2)(-6t) + (-6x^2 t)(2) = -18x^2 t. \\
 D_1 &= \frac{\partial V(x, 1)}{\partial x} U(x, 0) + \frac{\partial V(x, 0)}{\partial x} U(x, 1) \\
 &= (2x)(-12xt) + (2x)(-6xt) = -36x^2 t.
 \end{aligned}$$

$$E_1 = \frac{\partial^3 U(x, 1)}{\partial x^3} = 0.$$

Hence,

$$\begin{aligned}
 V(x, 2) &= -(RG)^{-1} \left[\frac{u}{s} RG(C_1) \right] - (RG)^{-1} \left[\frac{u}{s} RG(D_1) \right] - (RG)^{-1} \left[\frac{u}{s} RG(E_1) \right] \\
 &= -(RG)^{-1} \left[\frac{u}{s} RG(-18x^2 t) \right] - (RG)^{-1} \left[\frac{u}{s} RG(-36x^2 t) \right] - (RG)^{-1} \left[\frac{u}{s} RG(0) \right]
 \end{aligned}$$

$$\begin{aligned}
&= -(RG)^{-1} \left[-54x^2 \frac{u^2}{s^3} \right] \\
&= -54x^2 \frac{t^2}{2} = 27x^2 t^2.
\end{aligned}$$

Similarly, we can get $V(x,3), V(x,4), \dots$,

Finally, the solution $V(x,t)$ takes the form

$$\begin{aligned}
V(x,t) &= V(x,0) + V(x,1) + V(x,2) + \dots \\
&= x^2 - 6x^2 t + 27x^2 t^2 - 108x^2 t^3 + \dots + (-1)^n 3^n (n+1)x^2 t^n + \dots \\
&= \sum_{k=0}^n (-1)^k 3^k (k+1)x^2 t^k = \frac{x^2}{(1+3t)}.
\end{aligned} \tag{10}$$

From (9) and (10), we can see that the GRTM coupled with PRDTM gives also the exact solution for example 5.3.

6. Conclusion

In this paper, an integral transform method (Ramadan group transform) coupled with projected differential transform method (PDTM) have been successfully employed to obtain analytic solution for various types of partial differential equations. The method is simple, effective, efficient and easy to use where the main benefit of it is to offer the analytical approximation. In many cases the method the exact solution can be obtained in a rapid convergent series. Further study and full investigation for this new integral transform is under preparation and ready to be submitted. Also, to make this study more trustworthy and meaningful the definition of double Ramadan group is investigated and applied to many partial differential and integro-differential equations are solved. This contribution is submitted.

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