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On Vibration of Three-Layered Cylindrical Shell with Functionally Graded Middle Layer

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Abstract: In the current analysis vibration characteristics of a cylindrical shell composed of three layers are examined. Vibration of cylindrical shells is accomplished for their involvement in various areas of engineering and technology. Shell vibration behavior depends upon on different geometrical material parameters and material parameters. They provide the maximum stability of a physical system. There is graduation distribution of constituent materials in functionally graded materials and is controlled by polynomial, exponential and trigonometric volume exponent fraction laws. In the present study a cylindrical shell is composed of three layers whereas the middle layer consists of functionally graded material and the extreme layer are of isotropic nature. Material composition of the FG layer is governed by polynomial, exponential and trigonometric volume fraction exponent laws. Impact of these laws is examined on shell vibration frequencies for different physical parameters. Love's thin shell theory is adopted for shell motion equations. The vibration of cylindrical shells with FGM will be expressed by using the Raleigh-Ritz technique in this method. Three volume fraction laws are used to define the middle layer of tri-layer cylindrical shells. The Rayleigh-Ritz technique is applied to form the shell frequency equation which is solved by MATLAB software. The validity and accuracy of this method is investigated for a number of comparisons of numerical results.

Keywords: Component, Formatting, Style, Styling, Insert

1. Introduction

Cylindrical shells are essential components in the field of technology as well as that of engineering. Vibrations of cylindrical shells have been extensively studied for their simple geometrical designing. So a huge amount of research on them is seen in open literature. Egle *et al.* [1] examined free vibrations of orthogonally inflexible cylindrical shells where rigidness has been treated as distinct elements. Sharma *et al.* [2] investigated vibrations of cylindrical shells for clamped-free boundary conditions by using Rayleigh-Ritz technique. The vibration of cylindrical shells with intermediary supports was examined by Swaddiwudhpong *et al.* [3]. Vibrations of functionally graded (FG) cylindrical shells were investigated by Loy *et al.*[4] and Pardhan *et al.* [5] for various physical parameters and several boundary conditions. Li *et al.* [6] examined vibrations of circular cylindrical shells with FG materials middle layer for simply supported end conditions. They also used Love's approximation for strain and curvature-displacement relationships for shells The idea of tri layered cylindrical shells with intermediate layer of FGM was given by Batra [7] for studying axial buckling of cylindrical shells and they investigated this aspect of dynamical study of the shells. Bing *et al.* [8] examined vibration frequencies of thin walled cylindrical shells for different edge condition. Shao and Ma [9] investigate the vibration analysis of those cylindrical shells split into thin layer and used Fourier series expression method for SS-SS, C-C, C-F and C-SS boundary conditions. Naeem *et al.* [10] employed the Ritz formulation to investigate vibration of natural frequency characteristic of FG cylindrical shells. Naeem *et al.* [11] established the equation of FGM shells in eigenvalue expression to observe their frequencies.

In this study vibration characteristics of three layered cylindrical shell with FG middle layer are investigated. The frequencies analysis of two layers cylindrical shells was examined by Arshad *et al*. [12] in which one layer was FG layer and other layer was of homogeneous materials. Iqbal *et al*. [13] examined vibrations of FG cylindrical shells applying the wave propagation technique. The generalized differential quadrature method was applied to examine the vibration characteristics of FG materials cylindrical shells by Naeem *et al*. [14]. Sofiyev *et al*. [15] examined the non-linear free vibration of FG cylindrical shells attached to combine loads with various ends conditions and resting on elastic foundations. Vel [16] employed the elasticity solution technique to observe free and forced vibration of cylindrical shells. These shells were estimated by SS-SS boundary condition. Shah *et al*. [17] applied exponential volume fraction law to observe the cylindrical shell's vibration with FGM. Warburton *et al*. [18] investigated the appearance of frequency variations with the circuit wave and expressed the frequency in the form of shell energies. Vibration of spinning cylindrical shells was examined by Mehparvar [19]. The shells were constructed from FGM. They used the higher ordered theory for shell deformation with the use of energy Hamilton's principle to obtain the shell dynamical equations. The vibration of cylindrical shells which are containing FGM was observed by Lam *et al.* [20]. Their purpose was to check the effect of FGM on vibration characteristics of the shells. Their composition was maintained by volume fraction power law of distribution of materials in the radial direction. Yamanouchi *et al.*[21] and Koizumi [22] studied the structure and design of FGMs.

In this paper vibration of three layered cylindrical shells are analyzed for various shell parameters. The shell thickness consists of three layers where materials of the outer layers are of isotropic. The middle layer consists of FG materials. The shell problem has been written in the integral form by considering expressions of kinetic and strain energies for a cylindrical shell. The shell frequency equation is formed by applying the Raleigh-Ritz technique. The estimation of axial modal dependence is done by characteristic beam functions. These functions satisfy boundary conditions. Results are obtained for simply supported- simply supported, clamped-clamped, clamped- free and clamped-simply supported boundary conditions. Comparisons of results determined by this procedure are done with those found in literature to verify the validity and efficiency of this technique and accuracy of the results.

2. Theoretical Formulation

Figure 1, represents the geometry of a cylindrical shell. L, h, R, stand for its geometrical quantities viz.; length, thickness and mean radius respectively while E, ν and ρ designate Young's modulus, the Poisson ratio and the mass density respectively. The triplet (x,θ,z) defines an orthogonal coordinate system and they lie at the mid plane of the cylindrical shell. They describe the coordinates in the longitudinal, tangential and transverse directions respectively. The functions $u(x,\theta,z,t)$, $v(x,\theta,z,t)$ and $w(x,\theta,z,t)$ indicate for the longitudinal, tangential and transverse displacements from the mid surface of the shell.

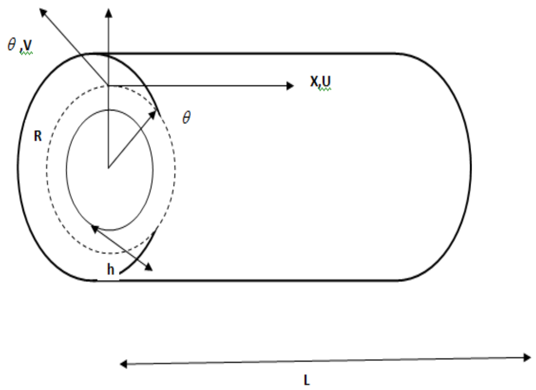

Figure 1. *Coordinate system and shell geometry.*

For a vibrating thin cylindrical shell, its strain energy, expressed by U is stated as Loy *et al.* [4]:

$$\begin{aligned} U = \frac{1}{2}\int_0^L\int_0^{2\pi} \{ & A_{11}e_1^2 + A_{22}e_2^2 + 2A_{12}e_1e_2 + A_{66}\gamma^2 \\ & +2B_{11}e_1k_1 + 2B_{12}e_1k_2 + 2B_{12}e_2k_1 + 2B_{22}e_2k_2 \\ & +4B_{66}\gamma\tau + D_{11}k_1^2 + D_{22}k_2^2 + 2D_{12}k_1k_2 + 4D_{66}\tau^2 \} R d\theta dx \end{aligned} \quad (1)$$

where the stress where e_1, e_2 and γ define the reference surface strains, k_1, k_2 and τ represent the surface curvatures and where A_{ij}, B_{ij} and D_{ij} $(i, j = 1,2 \text{ and } 6)$ are associated with the extensional , coupling and bending stiffness respectively and are stated as [4]:

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij}dz, B_{ij} = \int_{-h/2}^{h/2} Q_{ij}zdz, D_{ij} = \int_{-h/2}^{h/2} Q_{ij}z^2dz, \quad (2)$$

The reduced material stiffness $Q_{ij}(i, j = 1,2 \text{ and } 6)$ for isotropic materials are described as [4]::

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, Q_{12} = \frac{\nu E}{1-\nu^2}, Q_{66} = \frac{E}{2(1+\nu)} \quad (3)$$

for isotropic cylindrical shells the coupling stiffness considered equal zero and for the shells formed by FGM they considered non-zero. For the cylindrical shells which fabricated by FG materials their values depend on the material distribution. The negativity and positivity of coupling stiffness exist due to the irregularity of characteristics of materials at mid plane when reduced stiffness produced by physical properties of FG materials.

Also the kinetic energy of the cylindrical shell, denoted by T , is written as [4]

$$T = \frac{1}{2}\int_0^L\int_0^{2\pi} \rho_t \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] R d\theta dx \quad (4)$$

where t denotes the time variable and ρ_t represents the mass density per unit length and is written as

$$\rho_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dz \tag{5}$$

where ρ stand for the mass density.

2.1. Love's Shell Theory

Several shell theories have been found in the open literature. Kirchhoff's assumption is the basis for all shell theories. This assumption states that "Normal to the original mid-surface of a shell retains its normal position, suffer no change in length during deformation". Shell theory due Love is the pioneering one and all other modern theories have designed from it by modifying some physical terms. The formulas for strain and curvature–displacements are adopted from Love's shell theory to solve the present shell problem and are written as:

$$\{e_1, e_2, \gamma\} = \left\{\frac{\partial u}{\partial x}, \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right), \left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)\right\} \tag{6}$$

$$\{k_1, k_2, \tau\} = \left\{-\frac{\partial^2 w}{\partial x^2}, -\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right), -\frac{1}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)\right\} \tag{7}$$

These expressions for the surface strains e_1, e_2, and γ and the curvatures k_1, k_2, and τ from the relations (6) and (7) respectively are replaced into Equ.(1), the expression for strain energy, U attains the following form:

$$\begin{aligned} U = \frac{1}{2}\int_0^L\int_0^{2\pi} \{ & A_{11}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{A_{22}}{R^2}\left(\frac{\partial v}{\partial \theta} + w\right)^2 + \frac{2A_{12}}{R}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial \theta} + w\right) + A_{66}\left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)^2 - 2B_{11}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) - \frac{2B_{12}}{R^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial x}\right) \\ & - \frac{2B_{12}}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial x^2}\right) - \frac{2B_{22}}{R^3}\left(\frac{\partial v}{\partial \theta} + w\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) - \frac{4B_{66}}{R} \times \left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right)^2 \\ & \times \left(\frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta}\right)\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right) + D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right)^2 + \frac{2D_{12}}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right) + \frac{4D_{66}}{R^2}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right)^2 \Big\} R d\theta dx \end{aligned} \tag{8}$$

The Lagrange energy functional, symbolized by $\Pi = T - U$ for a cylindrical shell is described by the difference of its strain and kinetic energies as:

$$\Pi = T - U \tag{9}$$

The Raleigh-Ritz technique is used to examine the vibration of cylindrical shells. The deformation of cylindrical shells in longitudinal, tangential and transverse direction describe in the form of shell motion's equations with particular variables. Many kinds of mathematical functions are used to measure the axial modal dependence. The boundaries conditions of cylindrical shells are satisfied by them.

2.2. Modal Displacement Functions

The unidentified displacement functions $u(x,\theta,z)$, $v(x,\theta,z)$ and $w(x,\theta,z)$ showing deformations in the longitudinal, tangential and transverse directions are supposed in such shapes that the separation of the special and temporal variables is performed. This process is done by classical technique of separation of variables used for solving partial differential equations. The substitution of the presumed shapes of the modal displacement functions are made into the shell governing equations and a system of simultaneous equations is obtained in the vibration amplitude coefficient by the Rayleigh-Ritz method. The axial modal dependence related to the unknown functions is used to determine those functions which meet boundary conditions described for cylindrical shells. The following models for the modal deformation function are mentioned for axial, tangential and temporal variables:

$$\begin{aligned} u(x,\theta,z,t) &= AU(x)\sin n\theta \sin \omega t \\ v(x,\theta,z,t) &= BV(x)\sin n\theta \sin \omega t \\ w(x,\theta,z,t) &= CW(x)\sin n\theta \sin \omega t \end{aligned} \tag{10}$$

where

$$\begin{aligned} u(x,\theta,z,t) &= AU(x)\sin n\theta \sin \omega t \\ v(x,\theta,z,t) &= BV(x)\sin n\theta \sin \omega t \\ w(x,\theta,z,t) &= CW(x)\sin n\theta \sin \omega t \end{aligned}$$

and ω denotes the frequency of the cylindrical shell and n is the circumferential wave number. The coefficients A, B, C show the vibration amplitudes in the longitudinal, tangential and transverse directions respectively.

Substituting the above expressions of the shell energies into Equation (9), the new expression for the Lagrange functional is achieved as

$$\Pi = \frac{R\rho_t\omega^2}{2}\left[\int_0^L\int_0^{2\pi}\left\{A^2U^2(x)\sin^2 n\theta + B^2V^2(x)\cos^2 n\theta\right\}d\theta dx\right]\cos^2 \omega t - \frac{R\pi}{2}\left\{\int_0^L A_{11}A\left(\frac{dU}{dx}\right)^2 + \frac{A_{22}}{R^2}\left(-nBV(x) + CW(x)\right)^2\right.$$

$$+A_{66}\left(B\frac{dV}{dx}+\frac{n}{R}AU(x)\right)^2+\frac{2A_{12}A}{R}\left(-nBV(x)\frac{dV}{dx}+CW(x)\frac{dU}{dx}\right)$$
$$+2B_{11}\left(-AC\frac{dU}{dx}\frac{d^2W}{dx^2}\right)+\frac{2A_{12}A}{R^2}\left(-ACn^2\frac{dU}{dx}+nABV(x)\frac{dU}{dx}\right)$$
$$+2B_{12}\left(nBCV(x)\frac{d^2W}{dx^2}-C^2W(x)\frac{d^2W}{dx^2}\right)^2-\frac{2B_{22}}{R^3}(-n^2C^2W^2+n^3\ BCW(x)V(x)+nBCW(x)V(x)-n^2B^2V^2(x))$$
$$-\frac{4B_{66}}{R}\left(nBC\frac{dV}{dx}\frac{dW}{dx}-B^2\left(\frac{dV}{dx}\right)^2+\frac{n^2}{R}ACU(x)\frac{dW}{dx}-\frac{n}{R}ABU(x)\frac{dV}{dx}\right)+D_{11}C^2\left(\frac{d^2W}{dx^2}\right)^2+2D_{12}\left(-n^2C^2W(x)\frac{d^2W}{dx^2}+nBCV(x)\frac{d^2W}{dx^2}\right)-\frac{4}{R}D_{66}$$
$$\left(n^2C^2\left(\frac{dW}{dx}\right)^2+B^2\left(\frac{dV}{dx}\right)^2-2nBC\frac{dV}{dx}\frac{dW}{dx}\right)d\ \ dx.\sin^2\omega t \tag{11}$$

Applying the Rayleigh- Ritz method, the process of minimization is applied to the Lagrange functional Π and is partially differentiated with regard to the vibration amplitude coefficients A, B and C. So doing process of extremization of Π, the following required minimum value conditions are obtained:

$$\frac{\partial\Pi}{\partial A}=\frac{\partial\Pi}{\partial B}=\frac{\partial\Pi}{\partial C}=0 \tag{11}$$

2.3. Derivation of the Shell Frequency Equation

The point when terms of these compelling conditions adjusted in particular shape then shell recurrence mathematical statement is discovered. Three concurrent mathematical statements in A, B, C are acquired as:

$$C_{11}A+C_{12}B+C_{13}C=0 \tag{12}$$

$$C_{21}A+C_{22}B+C_{23}C=0 \tag{13}$$

$$C_{31}A+C_{32}B+C_{33}C=0 \tag{14}$$

where the coefficients C_{ij}'s $=(i,j=1,2,3)$ are some constants. The above equations can be written in the matrix form as

$$\begin{pmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{pmatrix}\begin{Bmatrix}A\\B\\C\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix} \tag{15}$$

This represents the frequency equation in the eigenvalue problem form. The condition of making the determinant of the matrix coefficients zero is applied for non-trivial solution for achieving the frequency equation.

2.4. Polynomial Volume Fraction Law

The properties of FG materials vary for temperature and they are originating in the field of high thermal condition. If the material property is denoted by P which is function of the absolute temperature T(K). Then Touloukian (1973) stated as:

$$P=P_0(P_{-1}T^{-1}+P_1T+P_2T^2+P_3T^3) \tag{16}$$

where the thermal coefficients are indicated by P0, P-1, P1, P2 and P3 while T indicates the temperature at absolute scale. The material properties of a FG constituent material for a cylindrical shell are functions of both temperature and their volume fractions. The succeeding material of a FG material is described as:

$$P=\sum_{j=1}^{K}P_jV_{f_j} \tag{17}$$

where the materials characteristics are mentioned by P_j's and the volume fraction of FGM denoted by V_{f_j}'s. Their sum always equal to one

$$\text{i.e.}\sum_{j=1}^{K}V_{f_j}=1 \tag{18}$$

V_f denotes the volume fraction of a FG material. It can be written as:

$$V_f=\left[\frac{z-h_2}{h_3-h_2}\right]^N \tag{19}$$

The thickness of cylindrical shell denoted by h and power-law exponent by N and its value always lie between zero and infinity. FGM are composition of two materials. For a FG cylindrical shell E, ν & ρ are expressed as:

$$E=(E_1-E_2)\left[\frac{z-h_2}{h_3-h_2}\right]^N+E_2 \tag{20}$$

$$\nu=(\nu_1-\nu_2)[(z-h_2)/(h_3-h_2)]^N+\nu_2 \tag{21}$$

$$\rho=(\rho_1-\rho_2)\left[\frac{z-h_2}{h_3-h_2}\right]^N+\rho_2 \tag{22}$$

where $z=-h/3,\ E=E_2,\ v=v_2$ denotes the materials used for M2 and $z=h/3,\ E=E_1,\ v=v_1$ describe the materials for M_1. Both the materials present on the inward and outward surfaces of cylindrical shells can change their materials characteristics by interchanging themselves. The cylindrical shells with FGM are usually in-homogeneous shell. When the thickness of a shell toward its radius ratio is less than 0.05 then the theory of classical thin-walled cylindrical shell is applicable.

2.5. Exponential Volume Fraction Law

Arshad et al. [10] modified the polynomial volume fraction law (20) and framed it in the exponential expression as:

$$V_j = 1-e^{-\left(z/h+0.5\right)^N}. \tag{23}$$

where $e=2.718\ \ldots$ is the usual natural base. Further formula is amended and a more general base $(b>0)$ is established and a new expression is written as:

$$V_j = 1-b^{-\left(z/h+0.5\right)N}, \tag{24}$$

Thus formulae for the effectual material properties: the effective Young's modulus E, the Poisson ratio E, v and the mass density ρ for a FG are written as:

$$\begin{aligned} E &= \left(E_1-E_2\right)\left(1-b^{-\left(z/h+0.5\right)^N}\right)+E_2, \\ v &= \left(v_1-v_2\right)\left(1-b^{-\left(z/h+0.5\right)^N}\right)+v_2, \\ \rho &= \left(\rho_1-\rho_2\right)\left(1-b^{-\left(z/h+0.5\right)^N}\right)+\rho_2, \end{aligned} \tag{25}$$

where $z=-h/2,\ E=E_2,\ v=v_2$ and $\rho=\rho_2$, when $z=h/2$, $E=\left(E_1-E_2\right)\left(1-b^{-1}\right)+E_2$, $v=\left(v_1-v_2\right)\left(1-b^{-1}\right)+v_2$, and $\rho=\left(\rho_1-\rho_2\right)\left(1-b^{-1}\right)+\rho_2$,

The above relations express M_2 present at the inward surface while M_1 at outward surface of the cylindrical shells.

2.6. Trigonometric Volume Fraction Law

This law obtained by making some changing in the formulae defines in (20) and (25) for cylindrical shell with FG layer related to M_1 and M_2 can be defined as:

$$V_{f1} = \sin^2\left[\left(z/h+0.5\right)^N\right] \tag{26}$$

$$V_{f2} = \cos^2\left[\left(z/h+0.5\right)^N\right] \tag{27}$$

where N is a positive real number. The conclude materials for this law can also express like other two laws for cylindrical shells with FG

$$\begin{aligned} E &= \left(E_1-E_2\right)\sin^2\left[\left(z/h+0.5\right)^N\right]+E_2 \\ v &= \left(v_1-v_2\right)\sin^2\left[\left(z/h+0.5\right)^N\right]+v_2 \\ \rho &= \left(\rho_1-\rho_2\right)\sin^2\left[\left(z/h+0.5\right)^N\right]+\rho_2 \end{aligned} \tag{28}$$

From formulae (30), when $v=v_2, \rho=\rho_2$ and when $z=h/2, E=\left(E_1-E_2\right)\left[\sin^2(1)\right]+E_2, v=\left(v_1-v_2\right)\left[\sin^2(1)\right]+v_2$, $\rho=\left(\rho_1-\rho_2\right)\left[\sin^2(1)\right]+\rho_2$, Thus at $z=-h/2$,

M_2 is attached at inward side but when $z=h/2$ then the characteristics material are obtained by both M_1 and M_2 materials present at outward surface of the cylindrical composed with FG material.

2.7. Material Stiffness for Three-Layered Cylindrical Shells

The thickness layer of the cylindrical shell is divided into three layers. Thicknesses of interior, intermediate and exterior layers are h_1, h_2 and h_3 respectively. For simplicity, thickness of each layer is of the thickness h/3. According to this configuration, the coefficients of extensional, coupling and bending stiffness $A_{ij,}$, B_{ij} and D_{ij} are modified as

Here *E, E_2 and E_2* are Young's moduli, N is power-law exponent and v_f volume fractions while v, v_1 and v_2 are Poisson ratios.

$$A_{11}=A_{22}=\frac{E(h_2-h_1)}{1-v_1^{\,2}}+\frac{E(h_3-h_2)}{1-v_f^{\,2}}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{E(h_4-h_3)}{1-v_2^{\,2}}$$

$$A_{12}=\frac{vE(h_2-h_1)}{1-v_1^{\,2}}+\frac{E(h_3-h_2)v_f}{1-v_f^{\,2}}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{vE(h_4-h_3)}{1-v_2^{\,2}}$$

$$A_{66}=\frac{E(h_2-h_1)}{2(1+v_1)}+\frac{(h_3-h_2)}{2(1+v_f)}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{E(h_4-h_3)}{2(1+v_2)}$$

$$B_{11}=B_{22}=\frac{E(h_2^2-h_1^2)}{2(1-v_1^2)}+\frac{1}{(1-v_f^2)}\left[\left(E_1-E_2\right)\left\{\frac{(h_3^2-h_2^2)}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E\left(h_3^2-h_2^2\right)}{2}\right]+\frac{E\left(h_4^2-h_3^2\right)}{(1-v_2^2)}$$

$$B_{12}=\frac{vE(h_2^2-h_1^2)}{2(1-v_1^2)}+\frac{v_f}{1-v_f^2}\left[\left(E_1-E_2\right)\left\{\frac{(h_3-h_2)^2}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E_2\left(h_3^2-h_2^2\right)}{2}\right]+\frac{vE\left(h_4^2-h_3^2\right)}{2(1-v_2^2)}$$

$$B_{66}=\frac{E(h_2^2-h_1^2)}{4(1+v_1^2)}+\frac{v_f}{2\left(1+v_f\right)}\left[\left(E_1-E_2\right)\left\{\frac{(h_3-h_2)^2}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E_2\left(h_3^2-h_2^2\right)}{2}\right]+\frac{E\left(h_4^2-h_3^2\right)}{4(1+v_2)}$$

$$D_{11}=D_{22}=\frac{E(h_2^3-h_1^3)}{3(1-v_1^2)}+\frac{1}{(1-v_f^2)}\left[\left(E_1-E_2\right)\left\{\frac{(h_3-h_2)^3}{N+3}+\frac{2h_2(h_3-h_2)}{N+2}+\frac{h_2^2(h_3-h_2)}{N+1}\right\}+\frac{E_2\left(h_3^3-h_2^3\right)}{6(1-v_f^2)}\right]+\frac{E\left(h_4^2-h_3^2\right)}{3(1-v_2^2)}$$

$$D_{12}=\frac{vE(h_2^3-h_1^3)}{3(1-v_1^2)}+\frac{v_f\left(E_1-E_2\right)}{(1-v_f^2)}\left[\left\{\frac{(h_3-h_2)^3}{N+3}+\frac{2h_2(h_3-h_2)^2}{N+2}+\frac{h_2^2(h_3-h_2)}{N+1}\right\}\right]+\frac{E_2\left(h_3^3-h_2^3\right)}{6(1-v_f^2)}+\frac{vE\left(h_4^2-h_3^2\right)}{3(1-v_2^2)}$$

$$D_{66}=\frac{E(h_2^3-h_1^3)}{6(1+v_1)}+\frac{\left(E_1-E_2\right)}{2(1+v_f)}\left[\frac{\left(h_3-h_2\right)^3}{N+3}+\frac{2h\left(h_3-h_2\right)^2}{N+2}+\frac{h_2^3\left(h_3-h_2\right)}{N+1}\right]+\frac{E_2\left((h_3^3-h_2^3)\right)}{6(1+v_f)}+\frac{E(h_4^3-h_3^3)}{6(1+v_2)}$$

3. Result and Discussion

The comparison of values of non-dimensional frequency parameters $\Omega=\omega R\sqrt{\left(1-v^2\right)\rho/E}$, for simply supported boundary conditions for homogeneous cylindrical shell with those of Loy et al. [4] is composed in Table 1. The present case was solved by the Raleigh-Ritz method while the frequency parameters in Loy et.al. [4] were obtained by the differential quadrature method. This comparison shows that the present results are nearly equal with each other. At n=2, the frequency parameter has the lowest value.

Table 1. *Comparison of frequency parameters* $\Omega=\omega R\sqrt{\left(1-v^2\right)\rho/E}$ *for a cylindrical shell with simply supported;-simply supported boundary conditions* $\left(m=1,L/R=20,h/R=0.01,v=0.3\right)$.

n	Loy et al. [4]	Present
1	0.016101	0.016101
2	0.009382	0.009363
3	0.022105	0.022085
4	0.042095	0.042075
5	0.068008	0.069788

Table 2. *Comparison of natural frequencies (Hz) for a simply supported-simply supported isotropic cylindrical shell (L=8in, h=0.1in, v=0.3,* $\left(m=1,L/R=20,h/R=0.01,v=0.3\right)$ $\left(m=1,L/R=20,h/R=0.01,v=0.3\right)$ ρ *=7.35×* 10^{-4} *Ibfs2 in-4, E=30×106Ibf in-2).*

n	N	Warburton[18]	Present
2	1	2946.8	2042.7
	2	5637.8	5631.9
	3	8935.3	8926.4
	4	11405	1139.3
	5	13245	13243.7
3	1	2199.3	2194.4
	2	4041.9	4031.2
	3	6620.0	6605.9
	4	9124.0	9108.4
	5	11357	11343.4

A comparison of the result of natural frequencies (Hz) for a cylindrical shell for simply supported-simply supported edge conditions is given with the results of Warburton [18] in the Table 2. These boundary conditions are applied at the both end points of the cylindrical shell. The half-wave axial numbers are taken to be *m* = 1, 2, 3, 4, 5, 6 and the circumferential wave numbers are taken *n*=2, 3. From the comparison it observed that these results are close to each other.

The results frequencies (Hz) of vibration cylindrical shells having FGM are obtained. These cylindrical shells consisof two types of FG material. Two materials: nickel and stainless steel are associated at inward and outward surfaces of a FG cylindrical shell of 1st Type. While in 2nd Type they interchange their positions. The outer surface denoted by M_1 and inner denoted by M_2. Natural frequencies (Hz) of 1st Type and 2nd Type cylindrical shells are composed in Table 3 and 4 respectively for the half-axial wave mode *m* =1. Geometric parameters are mentioned in the Tables. Polynomial fraction law regulates the material distributions in FGM. The power law exponents are taken as: *N*= 0.5, 1, 15.The present obtained frequencies and those of Iqbal *et al*. [13] are compared with each other. The shell frequencies have been evaluated by the Raleigh - Ritz method and wave propagation method was applied by Iqbal *et al*. [13] to obtain them. The condition which is stated at both the ends is simply supported-simply supported. So the compared results coincided with each other.

Table 3. *Natural frequencies (Hz) comparisons of 1st Type cylindrical shells having simply supported – simply supported end condition (m=1, L/R=20, h/R=0.002).*

	Iqbal *et al*. [13]			Present		
n	*N*=0.5	*N*=1	*N*=15	*N*=0.5	*N*=1	*N*=15
1	13.103	13.211	13.505	13.102	13.209	13.504
2	4.4382	4.4742	4.5759	4.4386	4.4742	4.5767
3	4.1152	4.1486	4.2451	4.1256	4.1578	4.2522
4	6.9754	7.0330	7.1943	6.9945	7.0497	7.2051
5	11.145	11.238	11.494	11.172	11.2611	11.5070

Table 4. *Natural frequencies (Hz) comparisons of 2nd Type cylindrical shells having simply supported-simply supported end condition. (m=1, L/R=20, h/R=0.002).*

	Iqbal et al. [13]			Present		
n	**N=0.5**	**N=1**	**N=15**	**N=0.5**	**N=1**	**N=15**
1	13.321	13.211	12.933	13.321	13.211	12.932
2	4.5168	4.480	4.3834	4.5195	4.4831	4.3858
3	4.1911	4.1569	4.0653	4.2014	4.1685	4.0788
4	7.0972	7.0384	6.8856	7.113	7.0563	6.9091
5	11.336	11.241	10.999	11.356	11.265	11.032

From the above comparisons, it is clear that the present numerical procedure is efficient and valid and yields accurate results.

Natural frequencies (Hz) for the present configurations of three layered cylindrical shells are furnished with variations depending on circumferential wave number, n for the half axial wave numbers, considering m=1. The end conditions considered here are simply supported – simply supported (SS-SS), clamped-clamped (C-C), clamped- free (C-F) and clamped-simply supported (C-SS). The three volume fraction laws: (i.) polynomial, (ii.) exponential and (iii.) trigonometric are applied to measure the material composition of FG layer.

Table 5. *Variation of natural frequencies (Hz) of 1st Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1) with polynomial fraction law.*

	N=0.5		N=1		N=5	
n	**SS-SS**	**C-C**	**SS-SS**	**C-C**	**SS-SS**	**C-C**
1	13.574	22.325	13.511	22.150	13.377	21.678
2	4.4803	7.5297	4.4653	7.4698	4.4052	7.3080
3	3.4889	4.1355	3.4437	4.0902	3.3525	3.9671
4	5.5134	4.4917	5.4206	4.4126	5.2341	4.1950
5	8.7339	6.5838	8.5821	6.4538	8.2779	6.0945
6	12.768	9.4979	12.545	9.3066	12.098	8.7771
7	17.558	13.023	17.251	12.759	16.636	12.029
8	23.092	17.113	22.687	16.766	21.878	15.806
9	29.366	21.756	28.851	21.315	27.822	20.093
10	36.379	26.948	35.741	26.401	34.466	24.888

Table 6. *Variation of natural frequencies (Hz) of 1st Type with C-F and C-SS boundary condition (L=20, h=0.002, R=1, m=1) with polynomial fraction law.*

	N=0.5		*N*=1		*N*=5	
n	**C-F**	**C-SS**	**C-F**	**C-SS**	**C-F**	**C-SS**
1	22.325	10.057	22.150	9.9785	21.811	9.8255
2	7.5302	3.3181	7.4704	3.2903	7.3540	3.2356
3	4.1365	2.5839	4.0915	2.5433	4.0028	2.4624
4	4.4927	4.0833	4.4139	4.0032	4.2580	3.8445
5	6.5845	6.4683	6.4548	6.3380	6.1980	6.0802
6	9.4984	9.4564	9.3072	9.2650	8.9292	8.8865
7	13.033	13.004	12.759	12.740	12.239	12.219
8	17.113	17.102	16.766	16.755	16.081	16.070
9	21.756	21.748	21.315	21.307	20.443	20.435
10	26.948	26.942	26.402	26.396	25.322	25.316

From the Tables 5 and 6, it is observed the natural frequencies (Hz) of 1st Type of cylindrical shells with four boundary conditions like SS-SS, C-C, C-F and C-SS for polynomial volume fraction law decreases when the value of power exponent N increases

The Tables 7 and 8, describe the natural Frequencies of 2nd Type of cylindrical shells with four boundary conditions like SS-SS, C-C, C-F and C-SS for polynomial volume fraction law. It is observed the frequencies increase when power exponent N increases

Table 7. *Variation of natural frequencies (Hz) of 2nd Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1) with polynomial fraction law.*

	N=0.5		N=1		N=5	
n	**SS-SS**	**C-C**	**SS-SS**	**C-C**	**SS-SS**	**C-C**
1	13.442	21.950	13.473	13.473	13.647	22.474
2	4.4217	7.3967	4.4318	4.4318	4.4890	7.5161
3	3.3347	3.9608	3.3418	3.3418	3.3823	4.0957
4	5.1848	4.0480	5.1956	5.1956	5.2558	4.2878
5	8.1969	5.8092	8.2140	8.2140	8.3084	6.2076
6	11.980	8.3458	12.005	12.005	12.143	8.9338
7	16.474	11.432	16.508	16.508	16.697	12.242
8	21.665	15.018	21.710	21.710	21.959	16.084
9	27.551	19.090	27.609	27.609	27.925	20.447
10	34.130	23.645	34.202	34.202	34.593	25.327

Table 8. *Variation of natural frequencies (Hz) of 2nd Type with C-F and C-SS boundary condition (L=20, h=0.002, R=1, m=1) with polynomial fraction law.*

	N=0.5		N=1		N=5	
n	**C-F**	**C-SS**	**C-F**	**C-SS**	**C-F**	**C-SS**
1	21.950	9.8997	22.121	9.9768	22.474	10.136
2	7.3962	3.2569	7.4546	3.2819	7.5757	3.3345
3	3.9597	2.4573	4.0036	2.4753	4.0947	2.5136
4	4.0469	3.8202	4.1258	3.8478	4.2868	3.9054
5	6.1575	6.0386	6.2023	6.0823	6.2946	6.1728
6	8.8670	8.8249	8.9313	8.8888	9.0639	9.0208
7	12.153	12.134	12.240	12.222	12.422	12.403
8	15.968	15.957	18.083	16.073	16.321	16.311
9	20.299	20.292	20.446	20.439	20.749	20.742
10	25.148	25.138	25.325	25.320	25.700	25.695

The variations of cylindrical shells having FG middle layer with SS-SS, C-C, C-F and C-SS boundary conditions for both types described in the Tables 9-12 for versus n for the half-axial wave mode, m=1 with exponential fraction law.

Table 9. *Variation of natural frequencies (Hz) of 1st Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	**SS-SS**	**C-C**	**SS-SS**	**C-C**	**SS-SS**	**C-C**
1	13.374	22.317	13.311	22.149	13.186	21.821
2	3.3600	7.5349	3.3438	7.4782	3.3104	7.3677
3	1.2446	4.0913	1.2388	4.0608	1.2245	4.0014
4	4.1349	4.3080	4.1166	4.2765	4.0800	4.2154
5	7.6059	6.2464	7.5722	6.2010	7.5057	6.1132
6	11.634	8.9922	11.582	8.9269	11.480	8.8007
7	16.288	12.323	16.216	12.234	16.074	12.061
8	21.602	16.192	21.505	16.075	21.317	15.848
9	27.591	20.585	27.468	20.436	27.228	20.147
10	34.264	25.498	34.112	25.313	33.814	24.956

Table 10. *Variation of natural frequencies (Hz) of 1st Type with C-F and C-SS boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	**C-F**	**C-SS**	**C-F**	**C-SS**	**C-F**	**C-SS**
1	22.235	9.9054	22.067	9.8309	21.740	9.6853
2	7.1317	2.4884	7.0792	2.4695	6.9698	2.4315

	N=0.5		N=1		N=5	
n	C-F	C-SS	C-F	C-SS	C-F	C-SS
3	3.1758	0.9217	3.1552	0.9149	3.0966	0.8994
4	3.4062	3.0623	3.3847	3.0402	3.3247	2.9968
5	5.6490	5.6330	5.6102	5.5922	5.5236	5.5130
6	8.5831	8.6162	8.5223	8.5538	8.3971	8.4328
7	12.026	12.063	11.940	11.975	11.768	11.806
8	15.966	15.998	15.851	15.882	15.624	15.658
9	20.407	20.434	20.260	20.286	19.972	19.999
10	25.355	25.376	25.171	25.192	24.814	24.836

It is observed from the Tables 8 and 9, the natural frequencies (Hz) of 1st Type of cylindrical shells with SS-SS, C-C, C-F and C-SS boundary conditions for exponential volume fraction law decreases when the value of power exponent *N* increases.

The natural frequencies (Hz) of 2nd Type of cylindrical shells with same mentioned above boundary conditions and volume fraction law catalogued in the Tables 11 and 12. It is observed the behaviour of natural frequencies is reverse of 2nd Type.

Table 11. *Variation of natural frequencies (Hz) of 2nd Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		*N*=1		*N*=5	
n	SS-SS	C-C	SS-SS	C-C	SS-SS	C-C
1	13.248	21.959	13.311	22.123	13.439	22.465
2	3.3257	7.4147	3.3418	7.4702	3.3760	7.5853
3	1.2283	4.0281	1.2340	4.0580	1.2487	4.1199
4	4.0992	4.2461	4.1172	4.2770	4.1548	4.3407
5	7.5417	6.1591	7.5751	6.2035	7.6433	6.2952
6	11.536	8.8671	11.587	8.9310	11.691	9.0628
7	16.151	12.152	16.223	12.240	16.369	12.420
8	21.421	15.267	21.516	16.083	21.709	16.320
9	27.360	20.299	27.481	20.446	27.728	20.747
10	33.978	25.144	34.128	25.326	34.434	25.699

Table 12. *Variation of natural frequencies (Hz) of 2nd Type with C-F and CC-SS boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		*N*=1		*N*=5	
n	C-F	C-SS	C-F	C-SS	C-F	C-SS
1	21.8769	9.7576	22.0404	9.8306	22.3811	9.9825
2	7.0099	2.4494	7.0614	2.4680	7.1716	2.5075
3	3.1066	0.9046	3.1269	0.9113	3.1777	0.9274
4	3.3397	3.0190	3.3610	3.0406	3.4134	3.0860
5	5.5596	5.5544	5.5978	5.5943	5.6818	5.6771
6	8.4568	8.4964	8.5165	8.5575	8.6430	8.6839
7	11.8545	11.8957	11.9390	11.9813	12.1157	12.1581
8	15.7410	15.7764	15.8537	15.8901	16.0880	16.1245
9	20.1214	20.1506	20.2658	20.2957	20.5650	20.5951
10	25.0006	25.0244	25.1803	25.2047	25.5519	25.5764

The variations of cylindrical shells having FG middle layer with SS-SS, C-C, C-F and C-SS boundary conditions for both types described in the Tables 13-16 for versus *n* for the half-axial wave mode, *m*=1 with trigonometric fraction law.

Table 13. *Variation of natural frequencies (Hz) of 1st Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	SS-SS	C-C	SS-SS	C-C	SS-SS	C-C
1	12.7887	21.1528	12.7325	20.9910	12.6077	20.6817
2	4.2153	7.0933	4.1950	7.0391	4.1552	6.9355
3	3.2995	3.8996	3.2840	3.8699	3.2528	3.8131
4	5.2247	4.2520	5.2002	4.2199	5.1521	4.1585
5	8.2760	6.2383	8.2372	6.1915	8.1615	6.1015
6	12.0972	8.9993	12.0405	8.9318	11.9302	8.8021
7	16.6337	12.3380	16.5556	12.2455	16.4042	12.0676
8	21.8714	16.2115	21.7714	16.0899	21.5724	15.8563
9	27.8155	20.6086	27.6849	20.4541	27.4319	20.1517
10	34.4549	25.5258	34.2951	25.3345	33.9817	24.9666

Table 14. *Variation of natural frequencies (Hz) of 1st Type with C-F and C-SS boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	C-F	C-SS	C-F	C-SS	C-F	C-SS
1	21.1529	9.4714	20.9911	9.3990	20.6817	9.2604
2	7.0938	3.1359	7.0396	3.0981	6.9357	3.0565
3	3.9006	2.4436	3.8710	2.4253	3.8136	2.3928
4	4.2529	3.8694	4.2210	3.8405	4.1590	3.7898
5	6.2389	6.1293	6.1922	6.0834	6.1018	6.0034
6	8.9998	8.9593	8.9323	8.8922	8.8024	8.7755
7	12.3383	12.3190	12.2458	12.2267	12.0678	12.0664
8	16.2117	16.2001	16.0902	16.0786	15.8564	15.8680
9	20.6088	20.6003	20.4543	20.4459	20.1572	20.1780
10	25.5260	25.5189	25.3346	25.3276	24.4967	24.9959

It is observed from the Tables 13 and 14, the natural frequencies (Hz) of 1st Type of cylindrical shells with SS-SS, C-C, C-F and C-SS boundary conditions for trigonometric volume fraction law decreases when the value of power exponent *N* increases.

The natural frequencies (Hz) of 2nd Type of cylindrical shells with same mentioned above boundary conditions and volume fraction law catalogued in the Tables 15 and 16. It is observed the behaviour of natural frequencies is reverse of 1st Type.

Table 15. *Variation of natural frequencies (Hz) of 2nd Type with SS-SS and C-C boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	SS-SS	C-C	SS-SS	C-C	SS-SS	C-C
1	12.678	20.895	12.753	21.054	12.896	21.378
2	4.1767	7.0094	4.2010	7.0626	4.2487	7.1711
3	3.2651	3.8502	3.2817	3.8793	3.3156	3.9387
4	5.1712	4.1898	5.1956	4.2212	5.2454	4.2853
5	8.1929	6.1432	8.2312	6.1890	8.3089	6.2829
6	11.976	8.8611	12.032	8.9272	12.145	9.0625
7	16.468	12.148	16.545	12.238	16.701	12.424
8	21.657	15.962	21.758	16.081	21.962	16.324
9	27.540	20.291	27.669	20.472	27.928	20.752
10	34.115	25.133	34.275	25.320	34.597	25.704

Table 16. *Variation of natural frequencies (Hz) of 2nd Type with C-F and CC-SS boundary condition (L=20, h=0.002, R=1, m=1).*

	N=0.5		N=1		N=5	
n	C-F	C-SS	C-F	C-SS	C-F	C-SS
1	20.895	9.3688	21.054	9.4399	21.378	9.5852
2	7.0089	3.0861	7.0621	3.1095	7.1709	3.1577
3	3.8492	2.4076	3.8783	2.4256	3.9382	2.4632
4	4.1889	3.8091	4.2202	3.8374	4.2848	3.8961
5	6.1426	6.0341	6.1883	6.0790	6.2825	6.1715
6	8.8607	8.8208	8.9267	8.8865	9.0623	9.0213
7	12.147	12.123	12.238	12.219	12.424	12.404
8	15.961	15.950	16.080	16.069	16.324	16.312
9	20.291	20.282	20.442	20.434	20.752	20.743
10	25.138	25.125	25.320	25.313	25.704	25.696

4. Conclusions

The vibration of cylindrical shells with FGM express by using the Raleigh-Ritz technique in this method. Three volume fraction laws are used to define the middle layer of tri-layer cylindrical shells. Two types of cylindrical shells are discussed in this method. The middle layer of cylindrical shell is FG which is composition of two materials Nickel and Stainless steel. At the inward surface of shell Stainless steel attached, while Nickel is attached at outward surface in 1^{st} Type of shells. The position of these materials will interchange in 2^{nd} Type. The results for simply-supported-simply supported, clamped-clamped, clamped-free and clamped- simply supported boundary conditions are obtained by this method. Following results are obtained by this present shell problem.

I. Circumferential wave number affect on the natural frequencies (Hz) of both Types of cylindrical shells. The frequencies increased and decreased by them.
II. Comparison of present obtained results with exponent power law for three volume fraction laws with the results of Loy *et al.*[4] and Naeem *et al.* [10-11] shows that they are good agreement with each other.
III. It observe that in 1^{st} Type of cylindrical shell frequency is increasing as N increase and in 2^{nd} Type it decreasing when N increase, due to interchanging the materials M_1 and M_2.
IV. The comparison of frequencies values of three volume fraction laws give the result that the frequency of 1^{st} Type cylindrical shell increasing by polynomial fraction law, while in 2^{nd} Type the frequency of cylindrical shells with clamped-clamped boundary condition increased by exponential law and other with polynomial fraction law. The comparison of variations of frequencies estimated that the recent method is valid and accurate. The obtained results are very close to previous result.

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