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Algebraic Points of Degree at Most 3 on the Affine Equation Curve $y^{11} = x^4(x-1)^4$

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Abstract: The quotients of Fermat curves $\mathcal{C}_{r,s}(p)$ are studied by Oumar SALL. Among these studies are the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 1$. Mamina COLY and Oumar SALL have explicitly determined the algebraic points of degree at most 3 on $\mathbb{Q}$ for the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 2$. Our work focuses on determining explicitly the algebraic points of degree at most 3 on $\mathbb{Q}$ on the curve $\mathcal{C}_{4,4}(11)$ which is a special case of Fermat quotient curves. Our study concerns the cases $\mathcal{C}_{r,s}(11)$ for $r = s = 4$. It seems that the finiteness of the Mordell-Weil group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$ is an essential condition. So to determine the algebraic points on the curve $\mathcal{C}_{4,4}(11)$ we need a finiteness of the Mordeill-Weill group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of rational points of the Jacobien is finite according to Faddev. Our note is in this framework. Our essential tools in this note are the Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the Jacobien of $\mathcal{C}_{4,4}(11)$ the Abel-Jacobi theorem and the study of linear systems on the curve $\mathcal{C}_{4,4}(11)$. The result obtained concerns some quotients of Fermat curves. Indeed, the curve $\mathcal{C}_{4,4}(11)$ which is the subject of our study, the set of algebraic points of degree at most 3 on $\mathbb{Q}$ has been determined in an explicit way, to achieve this we have determined the quadratic points on the curve $\mathcal{C}_{4,4}(11)$ on $\mathbb{Q}$ and the cubic points on the curve $\mathcal{C}_{4,4}(11)$ on $\mathbb{Q}$.

Keywords: Planes Curves, Degree of Algebraic Points, Jacobien

1. Introduction

Let $\mathcal{C}$ be a smooth algebraic curve defined on $\mathbb{Q}$. Let K be a field of numbers we note $\mathcal{C}(K)$ the set of points on $\mathcal{C}$ with coordinates in K, and $\bigcup_{[K\,:\,\mathbb{Q}]\leqslant d}\mathcal{C}(K)$ the set of points on $\mathcal{C}$ with coordinates in K of degree at most d on $\mathbb{Q}$. The degree of a point R of $\mathcal{C}$ algebraic on $\mathbb{Q}$ is defined as the degree of its defining field on $\mathbb{Q}$; in other words $deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$.

In this note we will focus on the curve $\mathcal{C}_{4,4}(11)$ with affine equation $y^{11} = x^4(x-1)^4$ which is a special case of quotients of Fermat curves $\mathcal{C}_{r,s}(p) : y^p = x^r(x-1)^s,\ 1 \leq r,\ s\,;\ r+s \leq p-1$ studied in [8-10]. The cases $\mathcal{C}_{r,s}(11)$ for $r = s = 2$ are studied in [2]. See [1, 3, 4, 11] for other explicit examples. Indeed, C corresponds to the curve $\mathcal{C}_{4,4}(11)$. The curves $\mathcal{C}_{r,s}(p)$ are quotients of F_p [6, 15].

We denote by $J_{4,4}(11)$ the Jacobien of $\mathcal{C}_{4,4}(11)$ and by $j(P)$ the class denoted $[P - P_\infty]$ of $P - P_\infty$, that is to say j is the Jacobien fold $\mathcal{C}_{4,4}(11) \longrightarrow J_{4,4}(11)$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the rational points of the Jacobien is finite [5, 6, 12, 13]. The curve $\mathcal{C}_{4,4}(11)$ in projective is $\mathcal{C}_{4,4}(11) : Y^{11} = X^4Z^7(X-Z)^4$.Let us note $P_0,\ P_1$ and P_∞ the points defined by: $P_0 = (0,0,1); P_1 = (1,0,1)$ and $P_\infty = (1,0,0)$. It follows from the work of *Gross-Rohrlich* in [6] ,that

$$\bigcup_{[K:\mathbb{Q}]\leqslant 2}\mathcal{C}_{1,1}(11)(K) = \left\{\left(\frac{1}{2} \pm \sqrt{y^{11}+\frac{1}{4}}, y\right) |y \in \mathbb{Q}\right\} \bigcup \{\ P_\infty\ \}$$

In this note we determine the set

$$\bigcup_{[K\,:\,\mathbb{Q}]\,\leqslant\,3} \mathcal{C}_{4,4}(11)(K)$$

Our main result is the following: *Theorem 1.1* The set of algebraic points of degree at most 3 on $\mathbb{Q}$ of the curve $\mathcal{C}_{4,4}(11)$ is given by:

$$\bigcup_{[K\,:\,\mathbb{Q}]\,\leqslant\,3} \mathcal{C}_{4,4}(11)(K) = \{P_0, P_1, P_\infty\} \cup S_0 \cup S_1$$

with

$$S_0 = \left\{\left(x,\ (\alpha x(x\ -\ 1))^{\frac{1}{3}}\right)\ \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x\ -\ 1)\ =\ \alpha^{11}\right\}$$

$$S_1 = \left\{\left(x,\ \left(\alpha(x(x\ -\ 1))^3\right)^{\frac{1}{8}}\right)\ \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x\ -\ 1)\ =\ \frac{1}{\alpha^{11}}\right\}$$

2. Auxiliary Results

For a divisor D on C, we note $\mathcal{L}(D)$ the $\bar{\mathbb{Q}}$-vector space of rational functions f defined on $\mathbb{Q}$ such that $f\ =\ 0$ or $div(f)\ \geq\ -D$; $l(D)$ denotes the $\bar{\mathbb{Q}}$-dimension of $\mathcal{L}(D)$.

Lemma 2.1 we have: $J_{4,4}(11)(\mathbb{Q})\ \cong\ \mathbb{Z}/11\mathbb{Z}$

Proof: According to Gross and Rohrlich ([6], $p.$ 219), we have: $J_{4,4}(11)(\mathbb{Q})_{torsion}\ \cong\ \mathbb{Z}/11\mathbb{Z}$, and According Faddeev [5], on a: $J_{4,4}(11)(\mathbb{Q})_{torsion}\ \cong\ J_{4,4}(11)(\mathbb{Q})$.

Lemma 2.2 $\mathcal{C}_{4,4}(11)\ :\ y^{11}\ =\ x^4(x\ -\ 1)^4$, we have:

(i)

$$div(x)\ =\ 11P_0\ -\ 11P_\infty;$$

$$div(x-1)\ =\ 11P_1\ -\ 11P_\infty;$$

$$div(y)\ =\ 4P_0\ +\ 4P_1\ -\ 8P_\infty.$$

(ii)

$$\mathcal{L}(P_\infty)\ = \langle 1 \rangle\,,$$

$$\mathcal{L}(2P_\infty)\ = \left\langle 1,\ \frac{y^3}{x(x\ -\ 1)} \right\rangle = \mathcal{L}(3P_\infty),$$

$$\mathcal{L}(4P_\infty)\ = \left\langle 1,\ \frac{y^3}{x(x\ -\ 1)},\ \frac{x^2(x\ -\ 1)^2}{y^5} \right\rangle = \mathcal{L}(5P_\infty),$$

$$\mathcal{L}(6P_\infty)\ \ =\ \ \left\langle 1,\ \frac{y^3}{x(x\ -\ 1)},\ \frac{x^2(x\ -\ 1)^2}{y^5},\ \frac{x(x\ -\ 1)}{y^2} \right\rangle = \mathcal{L}(7P_\infty),$$

$$\mathcal{L}(8P_\infty) = \left\langle 1,\ \frac{y^3}{x(x-1)},\ \frac{x^2(x-1)^2}{y^5},\ \frac{x(x-1)}{y^2},\ y \right\rangle = \mathcal{L}(9P_\infty),$$

$$\mathcal{L}(10P_\infty) = \left\langle 1,\ \frac{y^3}{x(x-1)},\ \frac{x^2(x-1)^2}{y^5},\ \frac{x(x-1)}{y^2},\ y,\ \frac{x^3(x-1)^3}{y^7} \right\rangle$$

$$\mathcal{L}(11P_\infty) = \left\langle 1,\ \frac{y^3}{x(x-1)},\ \frac{x^2(x-1)^2}{y^5},\ \frac{x(x-1)}{y^2},\ y,\ \frac{x^3(x-1)^3}{y^7},\ x, \right\rangle$$

$$\mathcal{L}(12P_\infty) = \left\langle 1,\ \frac{y^3}{x(x-1)},\ \frac{x^2(x-1)^2}{y^5},\ \frac{x(x-1)}{y^2},\ y,\ \frac{x^3(x-1)^3}{y^7},\ x,\ \frac{y^7}{x^2(x-1)^2} \right\rangle$$

Proof: Let x, y be the rational functions on $\mathbb{Q}$ given by: $x(X,\ Y,\ Z) = \frac{X}{Z}$ and $y(X,\ Y,\ Z) = \frac{Y}{Z}$, which allows to give the projective form of the curve

$$\mathcal{C}_{4,4}(11) : Y^{11} = X^4Z^7(X - Z)^4. \ (i)$$

(i) A)

$$div(x) = div(\frac{X}{Z}) = (X=0)\cdot\mathcal{C}_{4,4}(11) - (Z=0)\cdot\mathcal{C}_{4,4}(11)$$

a) For $X = 0$, we have $Y^{11} = 0$; pour $Z = 1$ we obtain the point $P_0 = (0,\ 0,\ 1)$ with an order of multiplicity equal to 11. Hence

$$(X = 0)\cdot\mathcal{C}_{4,4}(11) = 11(P_0). \tag{1}$$

b) The same goes for $Z = 0$, we have $Y^{11} = 0$;for $X = 1$ we obtain the point $P_\infty = (1,\ 0,\ 0)$ with an order of multiplicity equal to 11. Hence

$$(Z = 0)\cdot\mathcal{C}_{4,4}(11) = 11(P_\infty). \tag{2}$$

The relations (1) and (2) give

$$div(x) = 11(P_0) - 11(P_\infty).$$

B)

$$div(x-1) = div(\frac{X-Z}{Z}) = (X = Z)\cdot\mathcal{C}_{4,4}(11) - (Z = 0)\cdot\mathcal{C}_{4,4}(11)$$

a) For $X = Z$, the relation (i) give $Y^{11} = 0$.
We thus obtain the point $P_1 = (1,\ 0,\ 1)$ with an order multiplicity equal to 11. Hence

$$(X = Z)\cdot\mathcal{C}_{4,4}(11) = 11(P_1). \tag{3}$$

b) For $Z = 0$, we have $Y^{11} = 0$; for $X = 1$ We thus obtain the point $P_\infty = (1,\ 0,\ 0)$ with an order multiplicity equal to 11. Hence

$$(Z = 0)\cdot\mathcal{C}_{4,4}(11) = 11(P_\infty). \tag{4}$$

From relations 3 and 4 we deduce

$$div(x-1) \ = \ 11(P_0) \ - \ 11(P_\infty).$$

C)

$$div(y) \quad = \quad div(\frac{Y}{Z}) = (Y \ = \ 0) \cdot \mathcal{C}_{4,4} - (Z \ = \ 0) \cdot \mathcal{C}_{4,4}.$$

a) For $Y \ = \ 0$, we have $X^4(X \ - \ 1)^4 \ = \ 0$ when $Z \ = \ 1$; this give $X^4 \ = \ 0$ or $(X \ - \ 1)^4 \ = \ 0$. Hence

$$(Y \ = \ 0) \cdot \mathcal{C}_{4,4}(11) \ = \ 4(P_0) \ + \ 4(P_1). \tag{5}$$

b) the equation (i) can be written as: $Y^8 \ = \ X^4 Y^{-3} Z^7 (X \ - \ Z)^4$
Thus $Z \ = \ 0$ we $X \ = \ 1$, we have $Y^8 \ = \ 0$ we obtain the point $P_\infty \ = \ (1, \ 0, \ 0)$ with an order multiplicity equal to 8. Hence

$$(Z \ = \ 0) \cdot \mathcal{C}_{4,4} \ = \ 8(P_\infty) \tag{6}$$

The relations (5) et (6) lead to the fact that $div(y) \ = \ 4(P_0) \ + \ 4(P_1) \ - \ 8(P_\infty)$.

(ii) Resulte of (i).

Corollary 2.1 The following results are consequences of Lemma 2.

1) $11j(P_0) \ = \ 11j(P_1) \ = \ 0$

2) $4j(P_0) \ = \ -4j(P_1)$

So $j(P_0)$ and $j(P_1)$ generate the same group $J_{4,4}(11)(\mathbb{Q})$ isomorphic to $\mathbb{Z}/11\mathbb{Z}$.
Thus we have $J_{4,4}(11)(\mathbb{Q}) \ \cong \ \mathbb{Z}/11\mathbb{Z} \ = \ \{mj(P_0), \ 0 \ \leq \ m \ \leq \ 10\}$.

3. Demonstration of the Theorem

3.1. Quadratic Points on $\mathcal{C}_{4,4}(11)$

Let $R \ \in \ \mathcal{C}_{4,4}(11)(\mathbb{Q})$ with $[\mathbb{Q}(R) : \mathbb{Q}] \ = \ 2$. Let R_1 and R_2 be the conjugates of R in the Galois sense, and work with $t \ = \ [R_1 \ + \ R_2 \ - \ 2P_\infty]$ which is a point of $J_{4,4}(11)(\mathbb{Q}) \ = \ \{mj(P_0), \ 0 \ \leq \ m \ \leq \ 10\}$; so $t \ = \ mj(P_0)$ with $0 \ \leq \ m \ \leq \ 10$, thus

$$[R_1 \ + \ R_2 \ - \ 2P_\infty] \ = \ mj(P_0) \text{ with } 0 \ \leq \ m \ \leq \ 10 \qquad (k)$$

We notice that $R \ \notin \ \{P_0, \ P_1, \ P_\infty\}$.

1st case $m \ = \ 0$.

The formula (k) becomes $[R_1 \ + \ R_2 \ - \ 2P_\infty] \ = \ 0$.

There is a rational function f such that $div(f) \ = \ R_1 \ + \ R_2 \ - \ 2P_\infty$, so $f \ \in \ \mathcal{L}(2P_\infty)$. According to the lemma 2, we have $f \ = \ a \ + \ b\dfrac{y^3}{x(x \ - \ 1)}$ with $a \ \neq \ 0$ (otherwise one of R_i would be equal to P_0) and $b \ \neq \ 0$ (otherwise $\mathcal{L}(2P_\infty) \ = \ \mathcal{L}(P_\infty)$).

At points R_i we have:

$$a + b\frac{y^3}{x(x-1)} = 0$$

$$\Longleftrightarrow y^3 = -\frac{a}{b}x(x-1)$$

$$\Longleftrightarrow y \ = \ \left(-\frac{a}{b}x(x-1)\right)^{\frac{1}{3}}$$

On the other hand, we have:

$$y^{11} = x^4(x - 1)^4$$

$$\Longleftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x - 1))^{\frac{11}{3}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x-1))^4 (x(x-1))^{\frac{-1}{3}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} x^4(x-1)^4 (x(x-1))^{\frac{-1}{3}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x-1))^{\frac{-1}{3}} = 1$$

$$\Longleftrightarrow \left(-\frac{a}{b}\right)^{11} (x(x-1))^{-1} = 1$$

$$\Longleftrightarrow (x(x - 1)) = \left(-\frac{a}{b}\right)^{11}$$

$$\Longleftrightarrow (x(x-1)) = \alpha^{11}$$

We thus find a family of points:

$$\mathcal{S}_0 = \left\{ \left(x,\ (\alpha x(x - 1))^{\frac{1}{3}}\right) \mid \alpha \in \mathbb{Q}^* and\, x \text{ is the root of the equation } x(x - 1) = \alpha^{11} \right\}$$

2nd case $m = 1$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = j(P_0) = (4-3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0)$$

$$= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty]$$

Hence

$$[R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty, \text{so f} \in \mathcal{L}(9P_\infty).$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_1}\ f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $ord_{P_1}\ f \neq 4$ which is absurd. At points R_i, we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.

3rd case $m = 2$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 2j(P_0) = (4-2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0)$$

$$= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty, \text{so f} \in \mathcal{L}(8P_\infty).$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_1}\ f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $ord_{P_1}\ f \neq 4$ wich this absurd. At points R_i, we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.

4th case $m = 3$.

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 3j(P_0) = (4-1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0) = [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 + 4P_1 - 7P_\infty] = 0$$

There exists a rational function

f such that div$(f) = R_1 + R_2 + P_0 + 4P_1 - 7P_\infty$, so f $\in \mathcal{L}(7P_\infty)$.

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x - 1)} + c\frac{x^2(x - 1)^2}{y^5} + d\frac{x(x - 1)}{y^2}$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = 0$ which is absurd.

5th case $m = 4$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_1 - 6P_\infty] = 0$$

There exists a rational function f such that $div(f) = R_1 + R_2 + 4P_1 - 6P_\infty,$, so $f \in \mathcal{L}(6P_\infty)$.
According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x - 1)} + c\frac{x^2(x - 1)^2}{y^5} + d\frac{x(x - 1)}{y^2}$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ and therefore $f = 0$ which is absurd.

6th case $m = 5$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 5j(P_0) = (11 - 6)j(P_0) = 11j(P_0) - 6j(P_0)$$

At points R_i, we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + 6P_0 - 8P_\infty] = 0$$

There exists a rational function f such that $div(f) = R_1 + R_2 + 6P_0 - 8P_\infty$, so $f \in \mathcal{L}(8P_\infty)$.
According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x - 1)} + c\frac{x^2(x - 1)^2}{y^5} + d\frac{x(x - 1)}{y^2} + ey$$

We have $ord_{P_0} f = 6$ hence
$a = b = c = d = e = 0$ so $f = 0$ which contradicts the fact that $ord_{P_0} f = 6$ absurd.

7th case $m = 6$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 6j(P_0) = (8 - 2)j(P_0) = 8j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 8j(P_0) - 2j(P_0) = -8j(P_1) - 2j(P_0)$$

$$= [-8P_1 + 8P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 8P_1 + 10P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty, \text{so f} \in \mathcal{L}(12P_\infty)$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey + e_1\frac{x^3(x-1)^3}{y^7} + e_2x + e_3\frac{y^7}{x^2(x-1)^2}$$

We have $ord_{P_1} f = 8$, hence $a + e_2 = 0$ and $b = c = d = e = e_1 = e_3 = 0$; so $f = e_2(x-1)$ and therefore one of R_i should be equal to P_1 which is absurd.

8^{th} case $m = 7$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 7j(P_0) = (11-4)j(P_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_0 - 6P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 4P_0 - 6P_\infty, \text{so f} \in \mathcal{L}(6P_\infty).$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2}$$

We have $ord_{P_0} f = 4$ hence $a = b = c = d = 0$ so $f = 0$ which is absurd.

9^{th} case $m = 8$

The formula (k) becomes $[R_1 + R_2 - 2P_\infty] = 8j(P_0)$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$

Hence

$[R_1 + R_2 + 8P_1 - 10P_\infty] = 0$
There exists a rational function f such that $div(f) = R_1 + R_2 + 8P_1 - 10P_\infty$, so $f \in \mathcal{L}(10P_\infty)$.
According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey + e_1\frac{x^3(x-1)^3}{y^7}$$

We have $ord_{P_1} f = 8$ hence
$a = b = c = d = e = e_1 = 0$ so $f = 0$, which is absurd.

10th case $m = 9$
The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 9j(P_0) = (11-2)j(P_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 - 4P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 2P_0 - 4P_\infty, \text{so f} \in \mathcal{L}(4P_\infty).$$

According to the lemma 2, we have

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5}$$

We have $ord_{P_0} f = 2$ hence $a = b = 0$ so $f = c\frac{x^2(x-1)^2}{y^5}$ we have $c \neq 0$ otherwise $ord_{P_0} f \neq 2$ which is absurd. At points R_i, we have $0 = c\frac{x^2(x-1)^2}{y^5}$ hence $x^2(x-1)^2 = 0$ either $x = 0$ or $x = 1$ we find the points P_0 et P_1 which is absurd.

11th case $m = 10$
The formula (k) become

$$[R_1 + R_2 - 2P_\infty] = 10j(P_0) = (11-1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = -[P_0 - P_\infty] = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 - 3P_\infty] = 0$$

There exists a rational function

$$\text{f such that div}(f) = R_1 + R_2 + P_0 - 3P_\infty,$$

So
$f \in \mathcal{L}(3P_\infty)$ and as $\mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$
So one of R_i should be equal to P_∞ wich is absurd.

3.2. Cubic Points on $\mathcal{C}_{4,4}(11)$

Let $R \in \mathcal{C}_{4,4}(11)(\mathbb{Q})$ with $[\mathbb{Q}(R) : \mathbb{Q}] = 3$. Let R_1, R_2 and R_3 be the conjugates of R in the Galois sense, and work with $t = [R_1 + R_2 + R_3 - 3P_\infty]$ which is a point of $J_{4,4}(11)(\mathbb{Q}) = \{mj(P_0),\ 0 \le m \le 10\}$; so $t = mj(P_0)$ with $0 \le m \le 10$, thus

$$[R_1 + R_2 + R_3 - 3P_\infty] = mj(P_0) \text{ with } 0 \le m \le 10 \qquad (t)$$

We notice that $R \notin \{P_0,\ P_1,\ P_\infty\}$.
1st case $m = 0$.

The formula (t) becomes $[R_1 + R_2 + R_3 - 3P_\infty] = 0$.
There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 - 3P_\infty, \text{so f} \in \mathcal{L}(3P_\infty) \text{and as} \mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$$

So one of the R_i should be equal to P_∞ which is absurd.
2nd case $m = 1$
The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = j(P_0) = (4-3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0)$$

$$= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty, \text{so f} \in \mathcal{L}(10P_\infty).$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey + e_1\frac{x^3(x-1)^3}{y^7}$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey + e_1\dfrac{x^3(x-1)^3}{y^7}$; one of R_i should be equal to P_0 which

is absurd.
3rd case $m = 2$
The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 2j(P_0) = (4-2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0)$$

$$= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty, \text{so f} \in \mathcal{L}(9P_\infty) \text{ and as } \mathcal{L}(9P_\infty) = \mathcal{L}(8P_\infty)$$

Therefore one of R_i should be equal to P_∞ which is absurd.
4th case $m = 3$.
The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 3j(P_0) = (4-1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0)$$

$$= [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty, \text{so f} \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $ord_{P_1} f \neq 4$ which is absurd . At points R_i, we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.
5th case $m = 4$
The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] \quad = \quad 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 \ + \ R_2 \ + R_3 \ + 4P_1 - 7P_\infty] = 0$$

There exists a rational function f such that

$$div(f) \ = R_1 \ + \ R_2 \ + R_3 \ + 4P_1 - 7P_\infty, \text{so f} \ \in \ \mathcal{L}(7P_\infty)$$

According to the lemma 2, we have:

$$f \quad = \quad a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2}$$

We have $ord_{P_1}\ f \ = \ 4$ hence $a = b = c = d = 0$ and therefore one of $f = 0$ which is absurd.

6^{th} case $m \ = \ 5$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] \quad = \quad 5j(P_0) = (11-6)j(P_0) = 11j(P_0) - 6j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] \quad = \quad 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 \ + \ R_2 \ + R_3 \ + \ 6P_0 - \ 9P_\infty] = 0$$

There exists a rational function f such that

$$div(f) \ = R_1 \ + \ R_2 \ + \ R_3 \ + 6P_0 - 9P_\infty, \text{so f} \ \in \ \mathcal{L}(9P_\infty)$$

According to the lemma 2, we have:

$$f \quad = \quad a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_0}\ f \ = \ 6$ hence

$a = b = c = d = e = 0$ so $f = 0$ which contradicts the fact that $ord_{P_0} f = 6$ absurd.

7^{th} case $m \ = \ 6$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] \quad = \quad 6j(P_0) = (11-5)j(P_0) = 11j(P_0) - 5j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 5j(P_0) = - 5j(P_0) = [-5P_0 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 5P_0 - 8P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 5P_0 - 8P_\infty, \text{so f} \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$\begin{aligned} f &= a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} \\ &+ d\frac{x(x-1)}{y^2} + ey \end{aligned}$$

We have $ord_{P_0}\, f = 5$, hence $a = b = c = d = e = 0$; so $f = 0$ which contradicts the fact that $ord_{P_0} f = 5$ absurd.

8th case $m = 7$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 7j(P_0) = (11 - 4)(jP_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 4P_0 - 7P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 4P_0 - 7P_\infty, \text{so f} \in \mathcal{L}(7P_\infty) \text{ and as } \mathcal{L}(7P_\infty) = \mathcal{L}(6P_\infty)$$

So one of R_i should be equal to P_∞ which is absurd.

9th case $m = 8$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 8P_1 - 11P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 8P_1 - 11P_\infty, \text{so f} \in \mathcal{L}(11P_\infty)$$

According to the lemma 2, we have:

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} + d\frac{x(x-1)}{y^2} + ey + e_1\frac{x^3(x-1)^3}{y^7} + e_2x$$

We have $ord_{P_1}\ f = 8$, we must have $a + e_2 = 0$ and $b = c = d = e = e_1 = 0$; so $f = e_2(x-1)$ and therefore one of R_i should be equal to P_1 which is absurd.

10^{th} case $m = 9$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 9j(P_0) = (11-2)(jP_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 - 5P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 2P_0 - 5P_\infty, \text{so f} \in \mathcal{L}(5P_\infty)$$

According to the lemma 2, we have $f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5}$

We have $ord_{P_0}\ f = 2$ hence $a = b = 0$ so $f = c\frac{x^2(x-1)^2}{y^5}$ we have $c \neq 0$ otherwise $ord_{P_0}\ f \neq 2$ ce qui est absurde. At points R_i, we have $0 = c\frac{x^2(x-1)^2}{y^5}$ hence $x^2(x-1)^2 = 0$ either $x = 0$ or $x = 1$ we find the points P_0 and P_1 which is absurd.

11^{th} case $m = 10$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 10j(P_0) = (11-1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + P_0 - 4P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + P_0 - 4P_\infty, \text{so f} \in \mathcal{L}(4P_\infty)$$

And therefore

$$f = a + b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5}$$

$$ord_{P_0}\, f = 1 \Longrightarrow a = 0; \text{so f} = b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5}$$

$b \neq 0$ otherwise $ord_{P_0}\, f \neq 1$; which is absurd.
$c \neq 0$ sinon $f \in \mathcal{L}(3P_\infty)$; which is absurd.
At points R_i, we have

$$0 = b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} :$$

$$b\frac{y^3}{x(x-1)} + c\frac{x^2(x-1)^2}{y^5} = 0$$

$$\Longleftrightarrow y^8 = -\frac{c}{b}x^3(x-1)^3$$

$$\Longleftrightarrow y = \left(-\frac{c}{b}(x(x-1))^3\right)^{\frac{1}{8}}$$

On the other hand, we have:

$$y^{11} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{c}{b}(x(x-1)^3\right)^{\frac{11}{8}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{c}{b}\right)^{\frac{11}{8}} (x(x-1))^{\frac{33}{8}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{c}{b}\right)^{\frac{11}{8}} (x(x-1))^{\frac{32}{8}} (x(x-1))^{\frac{1}{8}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{c}{b}\right)^{\frac{11}{8}} x^4(x-1)^4 (x(x-1))^{\frac{1}{8}} = x^4(x-1)^4$$

$$\Longleftrightarrow \left(-\frac{c}{b}\right)^{\frac{11}{8}} (x(x-1))^{\frac{1}{8}} = 1$$

$$\Longleftrightarrow (-\frac{c}{b})^{11}x(x-1) = 1$$

$$\iff x(x-1) = (-\frac{b}{c})^{11}$$

$$\iff x(x - 1) = -(\frac{b}{c})^{11}$$

We thus find a family of points:

$$S_1 = \left\{ \left(x, \ \left(\alpha(x(x - 1))^3\right)^{\frac{1}{8}}\right) \mid \alpha \in \mathbb{Q}^* \text{and x is the root of the equation } x(x-1) = \frac{1}{\alpha^{11}} \right\}$$

4. Conclusion

Our note focuses on the determination of algebraic points on the curve $\mathcal{C}_{4,4}(11)$ of affine equation $y^{11} = x^4(x - 1)^4$. The curve $\mathcal{C}_{4,4}(11)$ is a special case of the quotients of Fermat curves. In this note an has explicitly determined the algebraic points of degree at most 3 on the curve $\mathcal{C}_{4,4}(11)$ on $\mathbb{Q}$. To do this we determined the quadratic points and the cubic points on $\mathcal{C}_{4,4}(11)$ on $\mathbb{Q}$.

It seems possible to determine explicitly the algebraic points of any given degree on the curve $\mathcal{C}_{4,4}(11)$.

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