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Some Characters of a Generalized Rational Difference Equation

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Abstract: Difference equations arise in many contexts in biological, economic and social sciences., can exhibit a complicated dynamical behavior, from stable equilibria to a bifurcating hierarchy of cycles. There are a lot of fascinating problems, which are often concerned with both mathematical aspects of the fine structure of the trajectories and practical applications. In this paper, we investigate the generalized rational difference equation, a kind of fractional linear maps with two delays. Sufficient conditions for the global asymptotic stability of the zero fixed point are given. For the positive equilibrium, we find the region of parameters in which the positive equilibrium is local asymptotic stable and attracts all positive solutions. As for general solutions, two specific and easy checked conditions on the initial values are obtained to guarantee corresponding solutions to be eventually positive. The upper or lower bound are also provided according to different parameters. Of particular interest for this generalized equation would be the existence of periodic solutions and their stabilities. We get the necessary and sufficient conditions for the existence of period two solutions depending on the combination of delay terms. In addition, the sufficient conditions for the existence of 2^r- and $2d-$periodic solutions are obtained too. In the end of the paper, we give examples to illustrate our results.

Keywords: Rational Difference Equation, Delay, Eventually Bounded, Eventually Positive, Periodic Solution

1. Introduction

Fractional linear maps play a key role in mathematical biology,population dynamics, and other research areas. The study of fractional linear map dates back to August Ferdinand Möbius (1790–1868). The Beverton-Holt map

$$f(x) = \frac{\mu x}{1+cx}, c = \frac{1-\mu}{K}$$

and the periodic Sigmoid Beverton-Holt equation

$$x_{n+1} = \frac{a_n x_n^{\delta_n}}{1+x_n^{\delta_n}},\ x_0 > 0,\ n = 0,1,2,\cdots$$

are examples of such a map ([1, 2])

The long-term behavior of solutions of these particular forms of the mapping has been studied widely when environmental fluctuations give rise to periodically varying carrying capacities K or both intrinsic growth rate μ and carrying capacity K change periodically with period p ([3, 4, 5]).

In this paper,we consider generalized rational difference equation with delay

$$x_{n+1} = \frac{Ax_{n-l} + Bx_{n-k}}{C + x_{n-k}}, n = 0,1,2,\cdots \tag{1}$$

where delays l and k are nonnegative integers satisfying $l \neq k$. A, B and C are real and positive parameters. Our discussions will be focused on the non-negative solutions because of its biological meanings. The famous Beverton-Holt equation is a special case when $A = 0$. We shall assume that all initial values are all nonnegative in this paper. The equation (1) can be rewritten as the following form

$$y_{n+1} = \frac{py_{n-l} + y_{n-k}}{q + y_{n-k}}, n = 0,1,2,\cdots \tag{2}$$

by a linear transformation $x_n = By_n$, where $p = \frac{A}{B} > 0$,

$q = \frac{C}{B} > 0$. So, we only consider the equation (2) from here on. Without loss of generality, we assume $l < k$ because all proofs for the case $l > k$ follow similar steps.

For the different k and l, the solutions of equation (2) changes significantly. Some special combinations of (l, k) have been studied in literatures. For example, R.Grove et al studied the equation (2) for $l = 1, k = 2$[6]. The case $l = 0$ was studied in [7]. E.Camouzis et al proved the global stabilities of the nonnegative equilibria for $l = 2$, $k = 3$ [8]. More related works and details can be found in [9, 10] for other special cases of (l, k).

This paper is organized as follows. In Section 2, we analyze the properties of equilibrium points. In Section 3, we consider the characters of general solutions. In Section 4, the existence and stability of periodic solutions are discussed. The last Section is a brief conclusion.

2. The Equilibrium Points

In this section, we review some known results about Eq. (2).

Any equilibrium point of Eq. (2) satisfies the equation

$$y = \frac{py + y}{q + y}.$$

Clearly, $y^* = 0$ is always an equilibrium point. Eq. (2) has a positive equilibrium point $\bar{y} = p+1-q$ if $q < p+1$. $y^* = 0$ will be unique nonnegative equilibrium point if $q \geq p + 1$. The stability of equilibrium $y^* = 0$ is given by the following theorem [11].

Theorem 2.1. For the Eq. (2), equilibrium $y^* = 0$ is globally asymptotically stable when $q \geq p + 1$, it is unstable when $q < p + 1$.

Proof. We only prove the local stability ([12])of zero equilibrium by definition. The proofs for the global stability and un-stability may be found in [11].

For $q \geq p + 1$ and any given ε positive, we choose $\delta = \varepsilon$. Then for any group of nonnegative initial values $\{y_{-k}, y_{-k+1}, \cdots, y_0\}$, we have

$$y_1 = \frac{py_{-l} + y_{-k}}{q + y_{-k}}.$$

Hence,

$$0 \leq y_1 < \frac{p+1}{q} \cdot \delta \leq \delta = \varepsilon \ \text{ for } y_{-i} < \delta, i = 0, 1, \cdots, k.$$

Similarly, we have

$$0 \leq y_n < \varepsilon, \ \ n = 1, 2, \cdots, k + 1.$$

This means that $y^* = 0$ is locally stable.

The solutions of Eq. (2) can be classified into two classes. The first class includes the solutions $\{y_n\}_{-k}^{\infty}$, which keep positive when n large enough for some nonnegative initial values. That is, there exists a positive integer N, such that $y_n > 0$ as $n \geq N$. Another class of solution $\{y_n\}_{-k}^{\infty}$ will have infinite many terms satisfying $y_n = 0$. For example, taking $l = 2, k = 5$, and initial values $\{y_{-5}, y_{-4} = 0, y_{-3}, y_{-2}, y_{-1} = 0, y_0\}$, we get $y_2 = y_5 = \cdots = y_{3m+2} = 0, \ \ m = 0, 1, 2, \cdots$ by a direct calculation.

Definition 2.1. A solution $\{y_n\}_{-k}^{\infty}$ of Eq. (2) is said to be eventually positive if there exists a positive integer N such that $y_n > 0$ for $n \geq N$. We will call a eventually positive solution a positive solution for simplicity.

Eq. (2) has unique positive equilibrium $\bar{y} = p + 1 - q$ when $q < p + 1$. The theorem 4.1 in [11] claims that $\bar{y}$ is globally asymptotically stable for $p - 1 < q < p + 1$. In fact, only (eventually) positive solutions approach to the $\bar{y}$ as $n \to \infty$. As a counter-example, we take $p = q$ and $l = 1, k = 3$. Then $\bar{y} = 1$. And Eq. (2) has a periodic solution with period 2, i.e. $\{y_n\}_{-k}^{\infty} = \{0, 1, 0, 1, \cdots\}$.

We rewrite this theorem and prove that all positive solutions approach the positive equilibrium $\bar{y} = 1$ for $p = q$. The proofs for other cases can be obtained by making a little change in [11].

Theorem 2.2. When $p - 1 < q < p + 1$, the positive equilibrium $\bar{y} = p + 1 - q$ of Eq. (2) is locally asymptotically stable, and all positive solutions converge to $p + 1 - q$.

Proof. The local asymptotical stability follows from Clark's Theorem [13]. We only prove that all positive solutions approach the positive equilibrium $\bar{y}$ for $p = q > 0$.

When $p = q$, we have

$$y_{n+1} - 1 = \frac{p(y_{n-l} - 1)}{p + y_{n-k}}.$$

For any positive solution, let

$$a = min\{y_{-k}, y_{-k+1}, \cdots, y_0, 1\} > 0.$$

Then $y_i \geq a$ for all $i = 1, 2, \cdots$, and

$$|y_{n+1} - 1| = \frac{p}{p + y_{n-k}} |y_{n-l} - 1| \leq \frac{p}{p + a} |y_{n-l} - 1|.$$

Therefore the subsequence $\{y_{n(l+1)+1}\}$ converges to finite limit 1 as $n \to \infty$ because of $a > 0$. Similarly, we get l subsequences

$$\{y_{n(l+1)+2}\}, \{y_{n(l+1)+3}\}, \cdots, \{y_{n(l+1)+l}\}, \{y_{n(l+1)}\},$$

which all converge to 1 as $n \to \infty$. Finally, we get

$$\lim_{n\to\infty} y_n = 1 = p + 1 - q.$$

3. Characters of General Solutions

In this section, we give some general characters of the solutions of Eq. (2). First we list a few of them from [11].

Lemma 3.1

(i) If $q > p$, then every solution of Eq. (2) is eventually bounded from above by the constant q/p.

(ii) If $q < p$, then every positive solution of Eq. (2) is eventually bounded from below by the constant q/p.

Lemma 3.2 If $q < p - 1$, then Eq. (2) has unbounded solutions for odd l and even k.

See [14] for the proof of Lemma 3.2.

The following example shows that Eq. (2) has non-trivial solutions with zero as lower bound.

Example 3.1. Suppose $q < p+1$, then Eq. (2) has at least l periodic solutions with period $l+1$ for $k = 2l+1$.

Proof. It is easy to check that

$$\{\underbrace{0,\cdots,0}_{m},\underbrace{p+1-q,\cdots,p+1-q}_{l+1-m}\}$$

are all $(l+1)-$periodic solutions for $m = 1,2,\cdots,l$. For example, we take $y_0 = y_{l+1} = 0$ and $y_i = y_{l+1+i} = \bar{y} = p+1-q,\ i = 1,2,\cdots,l$ for $m = 1$, then

$$y_{k+1} = \frac{py_{l+1} + y_0}{q + y_0} = 0;$$

and

$$y_{k+1+i} = \frac{py_{l+1+i} + y_i}{q + y_i} = \bar{y},\ \ i = 1,2,\cdots,l.$$

Generally, if taking $y_j = y_{l+1+j} = 0,\ j = 0,1,\cdots,m-1$, and $y_i = y_{l+1+i} = \bar{y} = p+1-q,\ i = m,m+1,\cdots,l$, then we have

$$y_{k+1+j} = \frac{py_{l+1+j} + y_j}{q + y_j} = 0,\ j = 0,1,\cdots,m-1;$$

and

$$y_{k+1+i} = \frac{py_{l+1+i} + y_i}{q + y_i} = \bar{y},\ \ i = m,m+1,\cdots,l.$$

Next theorem tells us when we have an eventually positive solution.

Theorem 3.1. Let $d = \gcd(l+1,k+1)$ is the greatest common divisor of $l+1$ and $k+1$, then followings are true.

(i) If $d = 1$, then all solutions with non-zero initial values are eventually positive.

(ii) If $d > 1$, then the solutions of Eq. (2) are eventually positive if initial values $\{y_{-k},y_{-k+1},\cdots,y_0\}$ contain at least d positive y_{α_i} $(i = 0,1,\cdots,d-1)$, where $\alpha_i = i \bmod d$ and $\alpha_i - \alpha_j \neq 0 \bmod d,\ i,j = 0,1,\cdots,d-1,\ i \neq j$.

Proof. (i) Suppose $d = 1$. Because of $l < k$, there exist positive integer $m \geq 1$ and positive integer $0 < r \leq l$, such that

$$k+1 = m(l+1) + r.$$

Without loss of the generality, we consider a solution with initial values

$$\{y_{-k},y_{-k+1},\cdots,y_0\} = \{0,0,\cdots,0,y_0 > 0\}.$$

Now, we try to find a positive integer N, such that $y_n > 0$ if $n \geq N$.

By Eq. (2), we have

$$y_{l+1} > 0,\ y_{2(l+1)} > 0,\cdots,\ y_{m(l+1)} > 0,\ y_{k+1} > 0$$

at once. That is, there are at least different $m+1$ items $y_j > 0$ $(1 \leq j \leq k+1)$ between y_1 and y_{k+1}. Using Eq. (2) again, we get at least different $2m+1$ items $y_j > 0$ $(k+2 \leq j \leq 2(k+1))$ between y_{k+2} and $y_{2(k+1)} = y_{2m(l+1)+2r}$ listed below

$$y_{(m+i)(l+1)} > 0,\ y_{k+1+i(l+1)} > 0,$$
$$y_{2(k+1)} > 0,\ i = 1,2,\cdots,m.$$

There are two possibilities now, $2r = l+1$ or $2r \neq l+1$.

If $2r = l+1$, then we get $k+1 = (2m+1)r$. So, $r = 1$ because $d = \gcd(l+1,k+1) = 1$. Therefore, we have $l = 1$ and $k+1 = 2m+1$. This means $y_n > 0$ for all $n \geq k+1$ because of $y_{k+1+i(l+1)} = y_{(m+i)(l+1)+1} > 0$.

If $2r \neq l+1$, we can get at least different $3m+1$ items $y_j > 0$ $(2(k+1)+1 \leq j \leq 3(k+1))$ between $y_{2(k+1)+1}$ and $y_{3(k+1)}$ listed below

$$\begin{cases} y_{(2m+i)(l+1)} > 0, \\ y_{k+1+(m+i)(l+1)} = y_{(2m+i)(l+1)+r} > 0, \\ y_{2(k+1)+i(l+1)} = y_{(2m+i)(l+1)+2r} > 0, \\ y_{3(k+1)} > 0, \end{cases} \qquad i = 1,\cdots,m.$$

Continuing in this way, we get, at s'th step, at least different $sm+1$ items $y_j > 0$ $((s-1)(k+1)+1 \leq j \leq s(k+1))$ between $y_{(s-1)(k+1)+1}$ and $y_{s(k+1)}$ listed below if $sr < l+1$

$$\begin{cases} y_j > 0,\ j = [(s-1)m+i](l+1)+tr, \\ y_{s(k+1)} > 0,\ s = 1,2,\cdots. \end{cases} \tag{3}$$

where $i = 1,2,\cdots,m; t = 0,1,2,\cdots,s-1$.

Note if $l+1 < sr < 2(l+1)$, we will get additional positive item $y_{(sm+1)(l+1)}$ at s'th step.

If $sr = l+1$, then $k+1 = (sm+1)r$. We then have $k+1 = sm+1$ because of $d = 1$. So $y_n > 0$ for all $n \geq (s-1)(k+1)$ according to the first equation of Eq. (3).

If we always have $sr \neq l+1$, there must be some positive integer s_1 such that $s_1 m + 1 \geq k+1$ because k is fixed. Therefore, $y_n > 0$ when $n \geq (s_1-1)(k+1)$.

(ii) Suppose $d > 1$. Denoting $l+1 = md, k+1 = rd$, where m and r are positive integers, $r > m \geq 1$. For a group of initial values $\{y_{-k},y_{-k+1},\cdots,y_0\}$ satisfying (ii), it can be divided into following d groups

$$\begin{cases} Y_0 &= \{y_{-k+d-1},y_{-k+2d-1},\cdots,y_0\}; \\ Y_1 &= \{y_{-k},y_{-k+d},\cdots,y_{-k+(r-1)d}\}; \\ \vdots & \\ Y_{d-1} &= \{y_{-k+d-2},y_{-k+2d-2},\cdots,y_{-k+rd-2}\}. \end{cases}$$

Clearly, $y_{\alpha_i} \in Y_i, i = 0,1,\cdots,d-1$.

For any positive integers s and t, we have

$$y_{\alpha_i+sd+t(k+1)} = \frac{py_{\alpha_i+(s-m)d+t(k+1)} + y_{\alpha_i+sd+(t-1)(k+1)}}{q + y_{\alpha_i+sd+(t-1)(k+1)}}.$$

This means that any item y_n of solution $\{y_n\}_{-k}^{\infty}$ depends on the items $y_{\bar{n}}$ only, where $\bar{n} < n$ and $n - \bar{n} = 0 \bmod d$.

Let $z_s = y_{\alpha_i+sd+t(k+1)}$, then

$$z_{s+1} = \frac{pz_{s-(m-1)} + z_{s-(r-1)}}{q + z_{s-(r-1)}}. \quad (4)$$

By (i), any solution of Eq. (4) with initial values $\{z_m, z_{m+1}, \cdots, z_{m+r-1}\}$

$$z_m = y_{\alpha_i+(l+1)+t(k+1)} > 0, \ \ i = 0, 1, \cdots, d-1$$

will be eventually positive because $\gcd(m, r) = 1$. So, the solution of Eq. (2) with d positive initial values y_{α_i} $(i = 0, 1, \cdots, d-1)$ must be eventually positive.

4. Periodic Solutions

In this section, we study the periodic solutions of Eq. (2) and their stabilities. Firstly, we give the necessary and sufficient conditions for the existence of 2−periodic solutions.

Theorem 4.1. The following statements are true.

(i) If l, k both are odd, then Eq. (2) has 2−periodic solutions if and only if $q < p + 1$.

(ii) If l is odd and k is even, then Eq. (2) has 2−periodic solutions if and only if $q = p - 1$. In this case, any 2−periodic solution $\{a, b, a, b, \cdots\}$ satisfies $b = \frac{a}{a-1}(a > 1, a \neq 2)$.

(iii) If l is even, then Eq. (2) has no any periodic solution with period 2.

Proof. (i) Suppose l, k both are odd.

It is easy to check that $\{0, p+1-q, 0, p+1-q, \cdots\}$ is a periodic orbit when $q < p + 1$.

Next, we assume that Eq. (2) has a periodic solution with period 2, denoted by $a, b, a, b\cdots$, where $a \neq b$. Then a and b must satisfy the following system

$$\begin{cases} a = \frac{pa+a}{q+a}, \\ b = \frac{pb+b}{q+b}. \end{cases}$$

So $q < p+1$ and $a = 0, b = p+1-q$ or $b = 0, a = p+1-q$.

(ii) Suppose l is odd and k is even.

If $q = p - 1$, then $p > 1$ by our assumption. Clearly, $\{p, \frac{p}{p-1}, p, \frac{p}{p-1}, \cdots\}$ is a 2−periodic solution when $p \neq 2$. We get the unique positive equilibrium point $\bar{y} = 2$ when $p = 2$. In fact, for any $a \in (1, 2)\bigcup(2, +\infty)$, $\{a, \frac{a}{a-1}, a, \frac{a}{a-1}, \cdots\}$ are all the 2−periodic solutions.

If Eq. (2) has a periodic solution with period 2, denoted by $\{a, b, a, b\cdots\}$, where $a \neq b$, then a and b must satisfy the following system

$$\begin{cases} a = \frac{pa+b}{q+b}, \\ b = \frac{pb+a}{q+a}. \end{cases}$$

So, we have $(a - b)(p - q - 1) = 0$. $q = p - 1$ follows at once.

In addition, if $q = p-1$ in above system, we have $ab = a+b$ or

$$b = \frac{a}{a-1}, \ \ a > 1.$$

This means that any 2−periodic solution of Eq. (2) must have the form $\{a, b, a, b, \cdots\}$, where $a > 1, a \neq 2, b = \frac{a}{a-1} > 1$.

(iii) Suppose l is even, then there exist two possibilities: k is odd or not.

Case I k is odd. If Eq. (2) has a 2−periodic solution $\{a, b, a, b\cdots\}$ $(a \neq b)$, then

$$\begin{cases} a = \frac{pb+a}{q+a}, \\ b = \frac{pa+b}{q+b}. \end{cases}$$

We get $a+b = 1-p-q$ at once through above system. So, $p+q < 1$ and both a and b must satisfy the following quadric equation

$$x^2 + (p+q-1)x - p(1-p-q) = 0. \quad (5)$$

The equation (5) has one positive root at most. This implies that Eq. (2) does not have any periodic solution with period 2.

Case II l and k are both even. Same as in the case I, for any 2−periodic solution $\{a, b, a, b\cdots\}$ $(a \neq b)$, we have

$$\begin{cases} a = \frac{pb+b}{q+b}, \\ b = \frac{pa+a}{q+a}. \end{cases}$$

$p + q + 1 = 0$ follows, which is impossible.

According to the Theorem 4.1, we have following corollary.

Corollary 4.1. If both l and k are odd, then Eq. (2) has unique 2−periodic solution $\{0, p+1-q, 0, p+1-q, \cdots\}$ when $q < p + 1$. If l is odd but k is even, then for any $a > 1, a \neq 2$, $\{a, \frac{a}{a-1}, a, \frac{a}{a-1}, \cdots\}$ always is a periodic solution of Eq. (2) with period 2 for $q = p - 1$.

Next we consider the stability of 2−periodic solutions of Eq. (2).

Theorem 4.2. The 2−periodic solution of the equation(2) is unstable if both l and k are odd.

Proof. If both l and k are odd, there must be two nonnegative positive integers s and t such that $l = 2s + 1, k = 2t + 1$. The unique 2−periodic solution $\{0, p+1-q, 0, p+1-q, \cdots\}$ is unstable because the un-stability of zero equilibrium of Eq. (2) by Theorem 2.1 and the odd items can be written as following

$$y_{2m-1} = \frac{py_{2m-2s-3} + y_{2m-2t-3}}{q + y_{2m-2t-3}}, \ \ m = 1, 2, \cdots.$$

Or

$$w_{m+1} = \frac{pw_{m-s} + w_{m-t}}{q + w_{m-t}}, \ \ m = 1, 2, \cdots, \quad (6)$$

where $w_m = y_{2m-1}$. In other words, the odd items of Eq. (2) satisfies Eq. (6), which is same as Eq. (2).

The following results say Eq. (2) may have 2^r−periodic solutions when $q = p-1$. *lemma 4.1* If $l = 4s-1, k = 4t+1$, then the equation(2) has 4−periodic solutions when $q = p-1$, where s and t are positive integers.

Proof. If $l = 4s - 1, k = 4t + 1$ and $q = p - 1$, then any solution of Eq. (2) can be expressed in the following forms according to the subscripts

$$\begin{cases} y_{2m-1} = \frac{py_{2m-1-4s}+y_{2m-1-(4t+2)}}{p-1+y_{2m-1-(4t+2)}}, \\ y_{2m} = \frac{py_{2m-4s}+y_{2m-(4t+2)}}{p-1+y_{2m-(4t+2)}}, \end{cases} \quad m = 1, 2, \cdots.$$

That is, the even items and odd items of the solutions are independent each other. Let $z_m = y_{2m}, w_m = y_{2m-1}$, then we have

$$\begin{cases} w_{m+1} = \frac{pw_{m-2s+1}+w_{m-2t}}{p-1+w_{m-2t}}, \\ z_{m+1} = \frac{pz_{m-2s+1}+z_{m-2t}}{p-1+z_{m-2t}}, \end{cases} \quad m = 0, 1, 2, \cdots.$$

Therefore, these two equations both have $2-$periodic solutions by Theorem 4.1. Suppose that $\{a_1, b_1, a_1, b_1, \cdots\}$ and $\{a_2, b_2, a_2, b_2, \cdots\}$ are $2-$periodic solutions of above equations about z_m and w_m respectively, where $b_i = \frac{a_i}{a_i-1}, a_i > 1, i = 1, 2$, and at least one of a_1 and a_2 is not equal to 2, then $\{a_1, a_2, b_1, b_2, a_1, a_2, b_1, b_2, \cdots\}$ is the $4-$periodic solution of original Eq. (2).

For example, taking $l = 3 = 4 - 1, k = 5 = 4 + 1$, then

$$\{2, 3, 2, \frac{3}{2}, 2, 3, 2, \frac{3}{2}, \cdots\},$$

$$\{3, 4, \frac{3}{2}, \frac{4}{3}, 3, 4, \frac{3}{2}, \frac{4}{3}, \cdots\},$$

and

$$\{3, 3, \frac{3}{2}, \frac{3}{2}, 3, 3, \frac{3}{2}, \frac{3}{2}, \cdots\}$$

are all periodic solutions of Eq. (2) with period 4.

Using the same method, we may prove the theorem below.

Theorem 4.3. If $q = p-1$ and $l = 2^r s-1, k = 2^r t+2^{r-1}-1$, then Eq. (2) has 2^r-periodic solutions, where s, t and r are positive integers.

Proof. Clearly, Eq. (2) has 2^r-periodic solutions for $r = 1$ and $r = 2$ by Theorem 4.1 and Lemma 4. We prove the theorem by induction on r $(r = 1, 2, 3, \cdots)$. Suppose the theorem is proved for positive integer r, then for the positive integer $r + 1$, we have

$$\begin{cases} y_{2m-1} = \frac{py_{2m-1-2^{r+1}s}+y_{2m-1-(2^{r+1}t+2^r)}}{p-1+y_{2m-1-(2^{r+1}t+2^r)}}, \\ y_{2m} = \frac{py_{2m-2^{r+1}s}+y_{2m-(2^{r+1}t+2^r)}}{p-1+y_{2m-(2^{r+1}t+2^r)}}, \end{cases} \quad m = 1, 2, \cdots.$$

Denoting the odd and even items by $z_m = y_{2m}, w_m = y_{2m-1}$ respectively, we get

$$\begin{cases} w_{m+1} = \frac{pw_{m-(2^r s-1)}+w_{m-(2^r t+2^{r-1}-1)}}{p-1+w_{m-(2^r t+2^{r-1}-1)}}, \\ z_{m+1} = \frac{pz_{m-(2^r s-1)}+z_{m-(2^r t+2^{r-1}-1)}}{p-1+z_{m-(2^r t+2^{r-1}-1)}}, \end{cases} \quad m = 0, 1, 2, \cdots.$$

By assumption, these two equations both have 2^r-periodic orbits $\{a_0, a_1, \cdots a_{2^r-1}\}$ and $\{b_0, b_1, \cdots, b_{2^r-1}\}$. Then $\{a_0, b_0, a_1, b_1, \cdots, a_{2^r-1}, b_{2^r-1}\}$ is the $2^{r+1}-$periodic solution of original Eq. (2).

Using Theorem 3.1 and Lemma 4, we get following interesting result.

Theorem 4.4. If $l = 2sd - 1, k = (2t + 1)d - 1$, then Eq. (2) has $2d-$periodic solutions for $q = p - 1$, where s, t and d are positive integers.

Proof. When $l = 2sd - 1, k = (2t + 1)d - 1$, the general item y_n of any solution can be expressed in the following forms according to the subscript n

$$\begin{cases} y_{md} = \frac{py_{(m-2s)d}+y_{(m-(2t+1))d}}{p-1+y_{(m-(2t+1))d}}, \\ y_{md-1} = \frac{py_{(m-2s)d-1}+y_{(m-(2t+1))d-1}}{p-1+y_{(m-(2t+1))d-1}}, \\ \vdots \\ y_{md-(d-1)} = \frac{py_{(m-2s)d-(d-1)}+y_{(m-(2t+1))d-(d-1)}}{p-1+y_{(m-(2t+1))d-(d-1)}}. \end{cases} \quad m = 1, \cdots.$$

That is, all items of solution are divided into d groups which are independent each other.

Let $z_m^{(i)} = y_{md-i}$ $(i = 0, 1, \cdots, d - 1)$, then we have

$$z_{m+1}^{(i)} = \frac{pz_{m-(2s-1)}^{(i)} + z_{m-2t}^{(i)}}{p - 1 + z_{m-2t}^{(i)}}, \ i = 0, 1, \cdots, d - 1.$$

Obviously, every one in above d equations has $2-$periodic solutions by Theorem 4.1. Suppose that $\{a_i, b_i, a_i, b_i, \cdots\}$ is a $2-$periodic solution about $z_m^{(i)}$, where $b_i = \frac{a_i}{a_i-1}, a_i > 1$ $(i = 0, 1, 2, \cdots, d - 1)$ and at least one of a_i is not equal to 2, then

$$\{a_0, \cdots, a_{d-1}, b_0, \cdots, b_{d-1}, a_0, \cdots, a_{d-1}, b_0, \cdots, b_{d-1}, \cdots\}$$

is a periodic solution of original Eq. (2) with period $2d$.

In fact, Theorem 4.3 is a special case of Theorem 4.4 for $d = 2^{r-1}$.

Example 4.1. If $l = 5, k = 8$, then Eq. (2) has periodic solutions with 6 for $q = p - 1$.

Proof. The conclusion follows because of $5 = 2\times 3-1, 8 = 3 \times 3 - 1$. For example,

$$\{2, 2, 3, 2, 2, \frac{3}{2}, 2, 2, 3, 2, 2, \frac{3}{2}, \cdots\}$$

and

$$\{2, 3, 4, 2, \frac{3}{2}, \frac{4}{3}, 2, 3, 4, 2, \frac{3}{2}, \frac{4}{3} \cdots\}$$

are two $6-$periodic solutions.

5. Conclusion and Discussion

We summarize what we get in this paper.

1. The zero equilibrium of Eq. (2) is globally asymptotically stable if $q \geq p + 1$; it's unstable if $q < p + 1$.
2. When $p-1 < q < p+1$, the unique positive equilibrium

$\bar{y} = p + 1 - q$ is locally asymptotically stable, and all positive solutions converge to it as $n \to +\infty$.

3. When $q = p - 1$, for any positive integer d, we could construct one equation such that it has periodic solutions with period $2d$.
4. Eq. (2) may have unbounded solutions for $0 < q < p - 1$.

Corollary 5.1. The followings are true.

(a) If $d = \gcd(l + 1, k + 1) = 1$, then any periodic solution of Eq. (2) must be positive.

(b) If $d = \gcd(l + 1, k + 1) > 1$, then Eq. (2) has $d - 1$ periodic solutions with period d for $q < p + 1$.

Note The example 3.1 is a special case of Cor.5.1 when $d = l + 1$.

Proof. (a) It follows from the Theorem 3.1 (i).

(b) Let $k + 1 = sd, l + 1 = td$, where s and t are positive integers. Consider a group of initial values

$$\{\underbrace{0,\cdots,0}_{d-1}, p+1-q, \underbrace{0,\cdots,0}_{d-1}, p+1-q, \cdots, \underbrace{0,\cdots,0}_{d-1}, p+1-q\}$$

That is, the initial values satisfy

$$y_i = \begin{cases} 0; & i = -k, -k+1, \cdots, 0, \\ & \text{but } i \neq -k + jd - 1,\ j = 1, 2, \cdots, s; \\ p + 1 - q; & \text{if } i = -k + jd - 1,\ j = 1, 2, \cdots, s. \end{cases}$$

We easily obtain

$$y_1 = y_2 = \cdots = y_{d-1} = 0,\ y_d = p + 1 - q.$$

So, it is a periodic solution with period d.

Similarly, we can prove that the solutions with the initial values

$$\{\underbrace{0,\cdots,0}_{d-r}, \underbrace{p+1-q,\cdots,p+1-q}_{r}, \cdots,$$
$$\underbrace{0,\cdots,0}_{d-r}, \underbrace{p+1-q,\cdots,p+1-q}_{r}\}$$

are all $d-$periodic solutions for $r = 1, 2, \cdots, d - 1$.

Problems We have several problems unsolved. For example, does there any positive odd-periodic solution exist? Is the positive equilibrium of Eq. (2) stable for $q \leq p-1$? How about the stability of general positive periodic solutions? etc.[6, 15]. These are of challenging.

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