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Stability Study of a Holling-II Type Model and Leslie-Gower Modified with Diffusion and Time Delays in Dimension 3

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Abstract: This current paper investigates a predator-prey model from Holling-II type and Leslie Gower modified with diffusion and two time delays in dimension three. Firstly, we demonstrate that its solutions are positive and globally bounded. Secondly, we study the local stability of six equilibria points of from one is located within the relevant domain. Under certain conditions, it reveals that among the equilibria points, some are locally stable. Finally, we focus on the global stability of the positive interior equilibrium point. We show that the global stability set out due to the lack of time delays is kept until a certain threshold value of time delays above which a change in the stability is observed. Thus, the global convergence analysis towards the positive interior equilibrium point demonstrate the importance and impacts of the time delay in the stability of our model.

Keywords: Holling-2, Leslie-Gower, Boundedness, Lyapunov's Functional, Equilibrium Point, Local Stability, Global Stability, Time Delay

1. Introduction

In the tropic network, the predator-prey link is characterized by the dynamic interaction between prey and predator populations. This interaction that can bind three and probably many species is the extension of the one that links two species [1]. This study deals with three-species food chain model. It is a dimension three model describing a population of preys U_1; that constitutes the only food of the predators' population U_2. This predator called intermediary is also the prey of another upper predator named super predator U_3.

The model takes into account the diffusion in predator-prey interactions and reflects the opportunity for each species present to move in a given space. For mathematical and simplification reasons, we select a limited open set Ω, from which we assume that the migration flows towards its boundary $\partial\Omega$ is null. Predators and preys density is also supposed null from the exterior of the chosen domain. We get the following model by adding the term of diffusion to the given model in the article [2]:

$$\begin{cases} \frac{\partial U_1(S,X)}{\partial S} = \delta_1 \Delta U_1(S,X) + (a_0 - b_0 U_1(S,X) \\ -\frac{v_0 U_2(S,X)}{U_1(S,X)+d_0}) U_1(S,X), \\ \frac{\partial U_2(S,X)}{\partial S} = \delta_2 \Delta U_2(S,X) + (-a_1 + \frac{v_1 U_1(S,X)}{U_1(S,X)+d_0} \\ -\frac{v_2 U_3(S,X)}{U_2(S,X)+d_2}) U_2(S,X), \\ \frac{\partial U_3(S,X)}{\partial S} = \delta_3 \Delta U_3(S,X) + (c_3 - \frac{v_3 U_3(S,X)}{U_2(S,X)+d_3}) U_3(S,X), \\ U_1(0,X) > 0, U_2(0,X) > 0, U_3(0,X) > 0, \\ \frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = \frac{\partial U_2}{\partial n} = 0,\ on\ ]0;+\infty[\times \partial\Omega \end{cases} \quad (1)$$

Where

$$X \in \Omega, S > 0$$

$(a_0, a_1, b_0, c_3, d_0, d_2, d_3, v_0, v_1, v_2, v_3) \in (\mathbb{R}_+^*)^{11}$ are ecological parameters [2],

$U_1(S,X)$, $U_2(S,X)$ and $U_3(S,X)$ respectively indicate not only the densities of both prey and intermediary predator but also that of the super predator at instant S and position X,

$\frac{dU_1(S,X)}{dS}$, $\frac{dU_2(S,X)}{dS}$ and $\frac{dU_3(S,X)}{dS}$ respectively designate the rate of prey, intermediaries predators and that of super predators increase at instant S and the position X depending on the following ecological parameters:

δ_1 is the prey diffusion coefficient U_1,
δ_2 is the predator diffusion coefficient U_2,
δ_3 is the super predator diffusion coefficient U_3,
Δ is the operator of Laplace.

In two dimensions, this schema was subject to many studies. There is a lot of articles about it [1, 3]. However, in dimension three, fewer works have been done. Aziz [7] studied a similar model to (1).

Nindjin and al [2] included the term $-b_1U_2(S)$ in the dynamics of the predator U_2. They studied the impact of certain time delays on the dynamic of these three species. This insertion permits to take in to account the internal competition between the members of this population especially in the research of food, procreation or occupation of the space.

Indeed, in the instantaneous case, most of them require the considered system to satisfy the so-called negative instantaneous diagonal feedback dominance condition. In the delayed system Lotka-Volterra-type, Kuang and Smith [8] showed that if, for every specy, the instantaneous intraspecific competition (instantaneous negative feedback) dominates the total competition due to delayed intraspecific competition and interspecific competition, then, the positive equilibrium point of the system remains globally and asymptotically stable. Most of the global stability or convergence results appearing so far for delayed ecological systems, require that the instantaneous negative feedbacks dominate both delayed feedback and interspecific interactions. Such requirement is rarely met in real systems when feedbacks are generally delayed. The model studied by Nindjin and al did not have any term of diffusion [2,7]. Therefore, the issue of mobility of the species in Ω has not been tackled. This approach is not always consistent with both ecological and biological realities. We introduce a term of diffusion to take into account the mobility of species to complement their works. The resulting model seems to be more realistic.

2. Presentation of the Model

Thus, this study is based on the following model:

$$\begin{cases} \frac{\partial U_1(S,X)}{\partial S} = \delta_1 \Delta U_1(S,X) + (a_0 \\ -b_0 U_1(S-\bar{r}_1, X) - \frac{v_0 U_2(S,X)}{U_1(S,X)+d_0}) U_1(S,X), \\ \frac{\partial U_2(S,X)}{\partial S} = \delta_2 \Delta U_2(S,X) + (-a_1 + \frac{v_1 U_1(S,X)}{U_1(S,X)+d_0} \\ -b_1 U_2(S-\bar{r}_2, X) - \frac{v_2 U_3(S,X)}{U_2(S,X)+d_2}) U_2(S,X), \\ \frac{\partial U_3(S,X)}{\partial S} = \delta_3 \Delta U_3(S,X) \\ +(c_3 - \frac{v_3 U_3(S,X)}{U_2(S,X)+d_3}) U_3(S,X), \\ U_1(\theta,.) = \phi_1(\theta,.), U_2(\theta,.) = \phi_2(\theta,.), U_3(\theta,.) = \phi_3(\theta,.), \\ U_1(0,.) > 0, U_2(0,.) > 0, U_3(0,.) > 0, \\ \phi = (\phi_1, \phi_2, \phi_3) \in C([-\bar{r}; 0] \times \bar{\Omega}; \mathbb{R}^3), \\ \frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = \frac{\partial U_2}{\partial n} = 0 \quad on\ ]0; +\infty[\times \partial\Omega, \end{cases} \tag{2}$$

Where $X \in \Omega,\ S > 0,\ \theta \in [-\bar{r}; 0]$ with

$$\bar{r} = \max\{\bar{r}_1; \bar{r}_2\}$$

In (2), the focus has not only been on the Laplacian operator which expresses the diffusion but especially, the time delays $\bar{r}_1$ and $\bar{r}_2$ which indicate respectively a time of recruitment for the prey U_1 and the intermediary predator U_2 [2]. In other words, the length of time necessary for these species to take part in the research of food, the procreation or occupation of space.

Furthermore, the insertion of both time delays and the term of diffusion is a relevant approach, because, it is more realistic and more interesting in the research for a better dynamic interaction comprehension between a predator and its prey. Considering the following variable changes:

$$t = a_0 S,\ x = (\frac{a_0}{\delta_1})^{\frac{1}{2}} X,\ W_1(t,x) = \frac{b_0}{a_0} U_1(T,X),\ W_2(t,x) = \frac{v_0 b_0}{a_0^2} U_2(T,X), W_3(t,x) = \frac{v_2 v_0 b_0}{a_0^3} U_3(T,X),$$

$a = \frac{d_0 b_0}{a_0}$, $b = \frac{a_1}{a_0}$, $\sigma_2 = \frac{\delta_2}{\delta_1}$, $\sigma_3 = \frac{\delta_3}{\delta_1}$, $c = \frac{v_1}{a_0}$, $e = \frac{a_0 b_1}{v_0 b_0}$, $d = \frac{d_2 v_0 b_0}{a_0^2}$, $p = \frac{c_3}{a_0}$, $q = \frac{v_3}{v_2}$, $s = \frac{d_3 v_0 b_0}{a_0^2}$, $r_1 = a_0 \bar{r}_1$ and $r_2 = a_0 \bar{r}_2$

we get the following completed model which is subject to our study:

$$\begin{cases} \frac{\partial W_1(t,x)}{\partial t} = \Delta W_1(t,x) + (1 - W_1(t - r_1, x) \\ -\frac{W_2(t,x)}{W_1(t,x)+a}) W_1(t,x), \\ \\ \frac{\partial W_2(t,x)}{\partial t} = \sigma_2 \Delta W_2(t,x) + (-b + \frac{c W_1(t,x)}{W_1(t,x)+a} \\ -e W_2(t - r_2, x) - \frac{W_3(t,x)}{W_2(t,x)+d}) W_2(t,x), \\ \\ \frac{\partial W_3(t,x)}{\partial t} = \sigma_3 \Delta W_3(t,x) \\ +(p - \frac{q W_3(t,x)}{W_2(t,x)+s}) W_3(t,x), \\ \\ W_1(\theta,x) = \phi_1(\theta,x), W_2(\theta,x) = \phi_2(\theta,x), \\ W_3(\theta,x) = \phi_3(\theta,x), \\ \\ W_1(0,x) > 0, W_2(0,x) > 0, W_3(0,x) > 0, \\ \phi = (\phi_1, \phi_2, \phi_3) \in C([-r;0] \times \bar{\Omega}; \mathbb{R}^3), \\ \\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_2}{\partial n} = 0 \quad on \ ]0; +\infty[\times \partial\Omega, \end{cases} \tag{3}$$

Where $x \in \Omega, t > 0, \theta \in [-r; 0]$ with

$$r = \max\{r_1; r_2\}$$

In this paper, our goal is to find out the natural, realistic and easily verifiable conditions under which, the global stability established in the instantaneous model remains the same. To achieve this aim, we set up an appropriated Lyapunov's functional. But, long before that, we showed the solutions boundedness by using methods employed in article [9]. Then, locally we analysed the equilibrium points of the system (3).

3. Global Boundedness of Solutions

Let us determine the sufficient conditions that ensure the global boundedness of the solutions of model (3). For that, for smallest and positive ε, let us define the following numbers:

$M_1 = e^{r_1} + \varepsilon, M_2 = \frac{c-b}{e} e^{(c-b)r_2} + \varepsilon, M_3 = \frac{p}{q}(M_2 + s) + \varepsilon,\ m_3 = \frac{ps}{q} - \varepsilon, m = K e^{(1-e^{Kr_1})Kr_1}$ $\quad m_1 = m - \varepsilon$ with $K = 1 - \frac{M_2}{a}$ and $m_2 = \frac{B}{e} e^{(1-e^{Br_2})Br_2} - \varepsilon$, with $B = -b - \frac{M_3}{d} + \frac{cm_1}{a+M_1}$.

Then, considering the domain

$$D = [m_1; M_1] \times [m_2; M_2] \times [m_3; M_3] \text{ of } \mathbb{R}^3, \tag{4}$$

Theorem 3.1 : If $c > b$, $e^{(c-b)r_2} < \frac{ae}{c-b}$ and $\frac{cm}{a+e^{r_1}} > \frac{p}{dq}(a+s)$ so, the system (3) is globally bounded and any solution of this system stays in the domain D.

In order to prove the theorem (3.1), let us state the following lemma.

Lemma 3.1 : Let us consider the system

$$\begin{cases} \frac{dv(t,x)}{dt} = \sigma \Delta v(t,x) + v(t,x) g(v(t-r,x)) \\ \frac{\partial v}{\partial n} = 0 \quad on \ [0; +\infty[\times \partial\Omega \\ v(s,x) \geq 0 \quad \forall s \in [-r; 0] \end{cases} \tag{5}$$

Where $t > 0,\ x \in \Omega$

If $v(0,x) \neq 0$, then, we have:

if $g(v) \leq \alpha(1 - \frac{1}{d} v)$ so, any solution v of (5) verifies $\text{limsup}_{t\to+\infty} \max_{\bar{\Omega}} v(t,.) \leq de^{\alpha r}$,

if $g(v) \geq \alpha(1 - \frac{1}{d} v)$ then, any solution v of (5) verifies $\text{liminf}_{t\to+\infty} \min_{\bar{\Omega}} v(t,.) \geq de^{A\alpha r}$ with $A = 1 - e^{\alpha r}$.

Proof : See the reference [9].

Now, let us prove theorem (3.1).

Proof : The two first equations of model (3) permit to lead to the two following inequations:

$$\frac{\partial W_2(t,x)}{\partial t} \leq \Delta W_1(t,x) + \left(1 - W_{12}(t - r_{12}, x)\right) W_1(t,x) \tag{6}$$

$$\frac{\partial W_2(t,x)}{\partial t} \leq \sigma_2 \Delta W_2(t,x) + \left(c - b - e W_2(t - r_2, x)\right) W_2(t,x) \tag{7}$$

According to lemma 3.1,

$\text{limsup}_{t\to+\infty} \max_{\bar{\Omega}} W_1(t,x) \leq e^{r_1}$ and $\text{limsup}_{t\to+\infty} \max_{x\in\bar{\Omega}} W_2(t,x) \leq \frac{c-b}{e} e^{(c-b)r_2}$.

Thus, $\forall \varepsilon > 0, \exists T_1 > 0, T_2 > 0 / \forall t > \max(T_1, T_2)$, $W_1(t,x) \leq e^{r_1} + \varepsilon$ and

$$W_2(t,x) \leq \frac{c-b}{e} e^{(c-b)r_2} + \varepsilon, \forall x \in \Omega$$

In that case, one gets : $W_1(t,x) \leq M_1$ and $W_2(t,x) \leq M_2$ with $M_2 > 0$ because $c > b$.

So, the third equation of (3) leads to:

$$\frac{\partial W_3(t,x)}{\partial t} \leq \sigma_3 \Delta W_3(t,x) + (p - \frac{q W_3(t,x)}{M_2 + s}) W_3(t,x) \tag{8}$$

By proceeding like previously: $\exists T_3 > 0 / \forall t > T_3$ $W_3(t,x) \leq M_3$. Consequently, $W_2(t,x) \leq M_2$, $W_1(t,x) \leq M_1$ and $W_3(t,x) \leq M_3, \forall t > \max\{T_1, T_2, T_3\}, \forall x \in \Omega$.

Let us seek the lowest values.

The first equation of (3) gives:

$$\frac{\partial W_1(t,x)}{\partial t} \geq \Delta W_1(t,x) + \left(1 - \frac{M_2}{a} - W_1(t-r_1,x)\right) W_1(t,x) \quad (9)$$

$$x \in \Omega, \;\; t > \max\{T_1 + r_1, T_2\}$$

By posing $K = 1 - \frac{M_2}{a}$ and applying the lemma 3.1 to (7), we show that $\exists\ T_4 > 0 / \forall t > T_4, W_1(t,x) \geq K e^{(1-e^{Kr_1})Kr_1} - \varepsilon\ \forall x \in \Omega$. It is clear that $K > 0$. So, for the smallest ε, $m_1 > 0$ and $W_1(t,x) \geq m_1$.

A similar approach permits to conclude that

$$\exists\ T_5 > 0\ /\ \forall t > T_5,\ m_3 \leq W_3(t,x),\ \forall x \in \Omega.$$

Taking into account the lowest values of W_3 and W_1, the second equation of (3) is:

$$\frac{\partial W_2(t,x)}{\partial t} \geq \sigma_2 \Delta W_2(t,x) + (B - eW_2(t-r_2,x))W_2(t,x)$$

Thus, $\exists\ T_6 > 0\ /\ \forall t > T_6, W_2(t,x) \geq \frac{B}{e} e^{(1-e^{Br_2})Br_2} - \varepsilon$. Then, $W_2(t,x) \geq m_2$.

Let us demonstrate that $m_2 > 0$. For that, we only have to prove that $B > 0$.

We have, $B = -b - \frac{\frac{p}{q}(M_2+s)+\varepsilon}{d} + \frac{cm_1}{a+e^{r_1}+\varepsilon}$. Considering the two first conditions of theorem 3.1 and by decaying ε toward zero, one gets:

$B > -c - \frac{pa}{dq} - \frac{ps}{dq} + \frac{cm}{a+e^{r_1}}$. Whereas

$$\frac{cm}{a+e^{r_1}} > \frac{p}{dq}(a+s), \text{ so, } B > 0$$

We conclude that $m_2 > 0$. Hence the result.

Remark 3.1 : When $r_1 = 0$ and $r_2 = 0$, we get the following values:

$M_1 = 1 + \varepsilon$, $M_2 = \frac{c-b}{e} + \varepsilon$, $M_3 = \frac{p}{q}(M_2 + s) + \varepsilon$, $m_3 = \frac{ps}{q} + \varepsilon$, $m_1 = 1 - \frac{M_2}{a} + \varepsilon$ and $m_2 = \frac{B}{e} + \varepsilon$ with

$$B = -b - \frac{M_3}{d} + \frac{cm_1}{a+M_1} \quad (10)$$

In that case, the conditions to have the boundedness are: $c > b$, $1 < \frac{ae}{c-b}$ and $\frac{cm}{a} > \frac{p}{dq}(a+s)$.

Remark 3.2 : In all the following work, we assume that the model is globally bounded.

4. The Equilibria Points

The system (3) has trivial equilibria points which are:

$$S_0 = (0,0,0), \qquad S_1 = (1,0,0), \qquad S_2 = \left(0,0,\frac{ps}{q}\right)$$

On the plan $W_1 = 0$, there is no equilibrium point.

On the plan $W_2 = 0$, one has only a trivial equilibrium point which is $S_3 = (1,0,\frac{ps}{q})$.

On the plan $W_3 = 0$, one has only a non trivial equilibrium point which is $S_4 = (W_1^*, (1-W_1^*)(W_1^*+a), 0)$ where W_1^* is eventually a positive root of

$P_1(x) = \gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0$ with

$$\gamma_0 = -a(b+e), \gamma_1 = c - b - ac, \gamma_2 = e(2a-1),\ \gamma_3 = e.$$

The polynomial P_1 admits at least a real root because it has an odd-degree.

Moreover, the product of these roots is $\frac{a(b+e)}{e} > 0$.

So, P_1 at least has a strictly positive root noted W_1^*. This must verify the condition.

$$\frac{ab}{c-b} < W_1^* < 1 \quad (11)$$

for S_4 to be a constant equilibrium point.

We have the following result relative of the interior equilibrum point.

Theorem 4.1 : If $\frac{\frac{ps}{q}+bd}{-6e+6c-4b-\frac{ps}{q}-bd} < a$ and $c > max(b, 3e + \frac{ps}{2q} + \frac{bd}{2})$, then, the system (3) admits a positive constant interior equilibrium point.

Proof : The system (3) admits a constant interior equilibrium point

$S_5 = (W_1^{**}, W_2^{**}, W_3^{**})$ if and only if $(W_1^{**}, W_2^{**}, W_3^{**})$ is the following system solution:

$$\begin{cases} 1 - W_1^{**} - \frac{W_2^{**}}{W_1^{**}+a} = 0, \\ -b + \frac{cW_1^{**}}{W_1^{**}+a} - eW_2^{**} - \frac{W_3^{**}}{W_2^{**}+d} = 0, \\ p - \frac{qW_3^{**}}{W_2^{**}+s} = 0. \end{cases} \quad (12)$$

By setting W_2^{**} and W_3^{**} according to W_1^{**} in (12), W_1^{**} becomes a root of the polynomial P_2.

$$P_2(x) = \gamma_5 x^5 + \gamma_4 x^4 + \gamma_3 x^3 + \gamma_2 x^2 + \gamma_1 x + \gamma_0$$

With

$$\gamma_0 = -\left[ea^3 + \left(de + \frac{p}{q}\right)a^2 + \frac{ps}{q}a + bda + ba^2\right],$$

$$\gamma_1 = -e(-2a^3 + 3a^2) - \left(de + \frac{p}{q}\right)(2a - a^2) - \frac{ps}{q} + cd + ac - bd - b(2a - a^2)$$

$$\gamma_2 = -e(a^3 - 6a^2 + 3a) - \left(de + \frac{p}{q} - b\right)(1 - 2a) + c(1 - a)$$

$$\gamma_3 = -e(3a^2 - 6a + 1) + \left(de + \frac{p}{q}\right) - c + b$$

$$\gamma_4 = -e(3a - 2)\gamma_5 = -e$$

As the degree of P_2 is odd, the polynomial P_2 admits at least a real root.
We get $P_2(0) = \gamma_0 < 0$ and

$$P_2(1) = \gamma_5 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 + \gamma_0 = a[-6e + 2c + 4(c - b)] - (a + 1)(\frac{ps}{q} + bd)$$

One has $P_2(1) > 0$ because $c > 3e$ and

$$\frac{a}{a+1} > \frac{\frac{ps}{q} + bd}{-6e + 2c + 4(c - b)}$$

As P_2 is continuous, under the conditions of theorem 4.1, at least,one of these roots which belong to $]0;1[$ noted W_1^{**}.
Hence S_5 exists and showed under the following form:

$$S_5 = (W_1^{**}, W_2^{**}, W_3^{**}) = (W_1^{**}, (1 - W_1^{**})(W_1^{**} + a), \tfrac{p}{q}(1 - W_1^{**})(W_1^{**} + a)) \tag{13}$$

5. Local Stability

In this subsection, we study the local stability in the neighborhood of S_k where k=0,...,5. We have two cases. One of them is the instantaneous system and the other is the system with delays.

5.1. Local Stability of the Instantaneous System

The model with no time delays looks as follows:

$$\begin{cases} \frac{\partial W_1(t,x)}{\partial t} = \Delta W_1(t,x) + (1 - W_1(t,x) \\ -\frac{W_2(t,x)}{W_1(t,x)+a})W_1(t,x), \\ \frac{\partial W_2(t,x)}{\partial t} = \sigma_2 \Delta W_2(t,x) + (-b + \frac{cW_1(t,x)}{W_1(t,x)+a} \\ -eW_2(t,x) - \frac{W_3(t,x)}{W_2(t,x)+d})W_2(t,x), \\ \frac{\partial W_3(t,x)}{\partial t} = \sigma_3 \Delta W_3(t,x) \\ +(p - \frac{qW_3(t,x)}{W_2(t,x)+s})W_3(t,x), \\ W_1(0,.) > 0, W_2(0,.) > 0, W_3(0,.) > 0, \\ \frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_2}{\partial n} = 0, on\ ]0;+\infty[\times \partial\Omega. \end{cases} \tag{14}$$

For that, let us consider $(\mu_i, \varphi_i)_{i=0}^{\infty}$ the set developed by the eigenvalue and eigenvector pairs of the operator $-\Delta$ on Ω with homogeneous Neumann boundary conditions such as $0 = \mu_0 < \mu_1 < \mu_2 <..$.

Let us note $X = \{(u,v,w) \in (C^2(\bar{\Omega}))^3 / \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0\}$. Given $E(\mu_i)$ the space of the eigenvectors corresponding to the eigenvalue μ_i for all

$$i = 0; 1; 2. ..,$$

$\{\phi_{ij}, j = 1; \ldots; \dim E(\mu_i)\}$ is an orthonormal basis of $E(\mu_i)$. $X_{ij} = \{c\phi_{ij}, c \in \mathbb{R}^3\}$.

X is a set that can be decomposed as a direct sum $\bigoplus_{i=0}^{\infty} X_i$, where

$$X_i = \bigoplus_{j=0}^{\dim E(\mu_i)} X_{ij}.$$

The operator resulting from the linearized system to the neighborhood of an equilibrium point is as follows:

$$\mathbb{K} = \begin{pmatrix} \Delta + A_{11}^{(k)} I_d & A_{12}^{(k)} I_d & A_{13}^{(k)} I_d \\ A_{21}^{(k)} I_d & \sigma_2 \Delta + A_{22}^{(k)} I_d & A_{23}^{(k)} I_d \\ A_{31}^{(k)} I_d & A_{32}^{(k)} I_d & \sigma_3 \Delta + A_{33}^{(k)} I_d \end{pmatrix} \tag{15}$$

With $A_{ml}^{(k)} = \frac{\partial f_m(S_k)}{\partial W_l}$, $k = 0, \ldots, 5$; $l = 1, \ldots, 3$ and $m = 1, \ldots, 3$. Δ Laplacian operator and I_d the identity of $C^2(\bar{\Omega})$.

Where

$$f_1(W_1(t,x), W_2(t,x), W_3(t,x)) = \left(1 - W_1(t,x) - \frac{W_2(t,x)}{W_1(t,x)+a}\right) W_1(t,x) \tag{16}$$

$$f_2(W_1(t,x), W_2(t,x), W_3(t,x)) = (-b + \frac{cW_1(t,x)}{W_1(t,x)+a} - eW_2(t,x) - \frac{W_3(t,x)}{W_2(t,x)+d})W_2(t,x) \tag{17}$$

$$f_3(W_1(t,x), W_2(t,x), W_3(t,x)) = (p - \frac{qW_3(t,x)}{W_2(t,x)+s})W_3(t,x). \tag{18}$$

For any fixed $i \geq 0$, X_i is invariant under $\mathbb{K}$. Thus, the $\mathbb{K}$ matrix in X_i becomes:

$$\begin{pmatrix} -\mu_i + A_{11}^{(k)} & A_{12}^{(k)} & A_{13}^{(k)} \\ A_{21}^{(k)} & -\sigma_2\mu_i + A_{22}^{(k)} & A_{23}^{(k)} \\ A_{31}^{(k)} & A_{32}^{(k)} & -\sigma_3\mu_i + A_{33}^{(k)} \end{pmatrix} \tag{19}$$

Our aim is to study the eigenvalues signs of (19) in order to establish the stability of the equilibria points S_k where $k = 1,\dots,5$

a. Stability of $S_0 = (0,0,0)$

One has : $A_{11}^{(0)} = 1,\ A_{12}^{(0)} = 0,\ A_{13}^{(0)} = 0,\ A_{21}^{(0)} = 0,\ A_{22}^{(0)} = -b,\ A_{23}^{(0)} = 0,\ A_{31}^{(0)} = 0,\ A_{32}^{(0)} = 0,\ A_{33}^{(0)} = p.$

The eigenvalues of (19) to the neighborhood of S_0 are $1 - \mu_i, -\sigma_2\mu_i - b$ and $-\sigma_3\mu_i + p$. We get:

If $\max\{\frac{p}{\sigma_3}, 1\} < \mu_i$ then, S_0 is stable,

Otherwise, S_0 is unstable.

b. Stability of $S_1 = (1,0,0)$

One has : $A_{11}^{(1)} = -1,\ A_{12}^{(1)} = \frac{-1}{1+a},\ A_{13}^{(1)} = 0,\ A_{21}^{(1)} = 0,\ A_{22}^{(1)} = -b + \frac{e}{1+a},\ A_{23}^{(1)} = 0,\ A_{31}^{(1)} = 0,\ A_{32}^{(1)} = 0,\ A_{33}^{(1)} = p$. The eigenvalues of (19) to the neighborhood of S_1 are $-1 - \mu_i, \sigma_2\mu_i - b + \frac{c}{1+a}$ and $-\sigma_3\mu_i + p$. We have:

If $\mu_i > \max\{\frac{1}{\sigma_2}(-b + \frac{c}{1+a}); \frac{p}{\sigma_3}\}$ then, S_1 is stable.

Otherwise, S_1 is unstable.

c. Stability of $S_2 = (0,0,\frac{ps}{q})$

We have: $A_{11}^{(2)} = 1,\ A_{12}^{(2)} = 0,\ A_{13}^{(2)} = 0,\ A_{21}^{(2)} = 0,\ A_{22}^{(2)} = -b - \frac{ps}{dq},\ A_{23}^{(2)} = 0,\ A_{31}^{(2)} = 0,\ A_{32}^{(2)} = 0,\ A_{33}^{(2)} = -p.$

The eigenvalues of (19) to the neighborhood of S_2 are $1 - \mu_i, -\sigma_2\mu_i - b - \frac{ps}{dq}$ and $-\sigma_3\mu_i - p$. We have:

If $1 < \mu_i$ then, S_2 is a stable node.

Otherwise, S_2 is unstable.

d. Stability of $S_3 = (1,0,\frac{ps}{q})$

we have: $A_{11}^{(3)} = -1,\ A_{12}^{(3)} = \frac{-1}{1+a},\ A_{13}^{(3)} = 0,\ A_{21}^{(3)} = 0,\ A_{22}^{(3)} = -b + \frac{c}{1+a} - \frac{ps}{dq},\ A_{23}^{(3)} = 0,\ A_{31}^{(3)} = 0,\ A_{32}^{(3)} = \frac{p^2}{q},\ A_{33}^{(3)} = -p.$

The eigenvalues of (19) to the neighborhood of S_3 are $-1 - \mu_i, -\sigma_2\mu_i + A_{22}^{(3)}$ and $-\sigma_3\mu_i - p$. We get:

If $\frac{A_{22}^{(3)}}{\sigma_2} < \mu_i$ so, S_3 is a stable node.

Otherwise, S_3 is unstable.

e. Stability of $S_4 = (W_1^*, W_2^*, 0)$ and $S_5 = (W_1^{**}, W_2^{**}, W_3^{**})$

The eigenvalues of (19) are the solutions of the following equation :

$$\lambda^3 + \alpha_2^{(k)}\lambda^2 + \alpha_1^{(k)}\lambda + \alpha_0^{(k)} = 0$$

With

$$\alpha_2^{(k)} = (1 + \sigma_2 + \sigma_3)\mu_i - (A_{22}^{(k)} + A_{11}^{(k)} + A_{33}^{(k)})$$

$$\alpha_1^{(k)} = (\sigma_2\sigma_3 + \sigma_2 + \sigma_3)\mu_i^2 - \left(\sigma_2 A_{11}^{(k)} + \sigma_3 A_{11}^{(k)} + A_{22}^{(k)} + A_{33}^{(k)} + \sigma_3 A_{22}^{(k)} + \sigma_2 A_{33}^{(k)}\right)\mu_i + A_{11}^{(k)}A_{22}^{(k)} + A_{11}^{(k)}A_{33}^{(k)} + A_{22}^{(k)}A_{33}^{(k)} - A_{23}^{(k)}A_{32}^{(k)} - A_{12}^{(k)}A_{21}^{(k)}$$

$$\alpha_0^{(k)} = \sigma_2\sigma_3\mu_i^3 - \left(A_{11}^{(k)}\sigma_2\sigma_3 + A_{22}^{(k)}\sigma_3 + A_{33}^{(k)}\sigma_2\right)\mu_i^2 + \left(A_{11}^{(k)}A_{22}^{(k)}\sigma_3 + A_{11}^{(k)}A_{33}^{(k)}\sigma_2 + A_{33}^{(k)}A_{22}^{(k)} - A_{23}^{(k)}A_{32}^{(k)} - A_{12}^{(k)}A_{21}^{(k)}\sigma_3\right)\mu_i - A_{11}^{(k)}A_{22}^{(k)}A_{33}^{(k)} + A_{11}^{(k)}A_{23}^{(k)}A_{32}^{(k)} + A_{21}^{(k)}A_{12}^{(k)}A_{33}^{(k)}$$

Theorem 5.1 : S_k is stable if and only if

$$\alpha_2^{(k)} > 0,\ \alpha_0^{(k)} > 0 \ \textit{and}\ \alpha_2^{(k)} \times \alpha_1^{(k)} > \alpha_0^{(k)} \ (k = 4; 5) \tag{20}$$

Proof : The eigenvalues of (19) are the solutions of the following equation :

$$\lambda^3 + \alpha_2^{(k)}\lambda^2 + \alpha_1^{(k)}\lambda + \alpha_0^{(k)} = 0 \tag{21}$$

Whereas according to Routh-Hurwitz criterion, this equation admits some solutions with a real negative part only if :

$$\alpha_2^{(k)} > 0,\ \alpha_0^{(k)} > 0 \text{ and } \alpha_2^{(k)} \times \alpha_1^{(k)} > \alpha_0^{(k)} \tag{22}$$

Hence we have the following result.

Remark 5.1 : To get $\alpha_2^{(5)} > 0$, $\alpha_0^{(5)} > 0$ and $\alpha_2^{(5)} \times \alpha_1^{(5)} > \alpha_0^{(5)}$, we only have $A_{11}^{(5)} < 0$ and $A_{22}^{(5)} < 0$.

We already know that

$$A_{11}^{(5)} = 1 - 2W_1^{**} - \frac{aW_2^{**}}{(W_1^{**}+a)^2} = \frac{W_1^{**}(-2W_1^{**}+1-a)}{W_1^{**}+a}$$

So, when $W_1^{**} > \frac{1-a}{2}$, one gets $A_{11}^{(5)} < 0$.

In a similar way, when $W_1^{**} < \frac{ab}{c-b}$, $A_{22}^{(5)} < 0$.

Consequently a sufficient condition to have the local stability of S_5 is : $\frac{1-a}{2} < W_1^{**} < \frac{ab}{c-b}$ provided that $\max(\frac{c-b}{c+b}, \frac{\frac{ps}{q}+bd}{-6e+6c-4b-\frac{ps}{q}-bd}) < a$.

5.2. Local Stability of the System with Time Delays

Let us note $(W_1^{(k)}, W_2^{(k)}, W_3^{(k)})$ the equilibria points components S_k with $k = 0;\ldots;5$ and let us pose $V_1(t,x) = W_1(t,x) - W_1^{(k)}$, $V_2(t,x) = W_2(t,x) - W_2^{(k)}$, and $V_3(t,x) = W_3(t,x) - W_3^{(k)}$

The linearized system to the neighborhood of the equilibria points S_k is for all $x \in \Omega$ and $t > 0$:

$$\begin{cases} \frac{\partial V_1(t,x)}{\partial t} = \Delta V_1(t,x) + A_{11}^{(k)} V_1(t,x) \\ +A_{12}^{(k)} V_2(t,x) + A_{14}^{(k)} V_1(t-r_1,x), \\ \frac{\partial V_2(t,x)}{\partial t} = \sigma_2 \Delta V_2(t,x) + A_{21}^{(k)} V_1(t,x) + A_{22}^{(k)} V_2(t,x) \\ +A_{23}^{(k)} V_3(t,x) + A_{24}^{(k)} V_2(t-r_2,x), \\ \frac{\partial V_3(t,x)}{\partial t} = \sigma_3 \Delta V_3(t,x) + A_{32}^{(k)} V_2(t,x) + A_{33}^{(k)} V_3(t,x), \end{cases} \tag{23}$$

with $A_{11}^{(k)} = 1 - W_1^{(k)} - \frac{aW_2^{(k)}}{(W_1^{(k)}+a)^2}$, $A_{12}^{(k)} = -\frac{W_1^{(k)}}{W_1^{(k)}+a}$, $A_{14}^{(k)} = -W_1^{(k)}$, $A_{21}^{(k)} = \frac{caW_2^{(k)}}{(W_1^{(k)}+a)^2}$, $A_{22}^{(k)} = -b + \frac{cW_1^{(k)}}{W_1^{(k)}+a} - eW_2^{(k)} - \frac{dW_3^{(k)}}{(W_2^{(k)}+d)^2}$, $A_{23}^{(k)} = -\frac{W_2^{(k)}}{W_2^{(k)}+d}$, $A_{24}^{(k)} = -eW_2^{(k)}$, $A_{32}^{(k)} = \frac{q(W_3^{(k)})^2}{(W_2^{(k)}+s)^2}$, $A_{33}^{(k)} = p - \frac{2qW_3^{(k)}}{W_2^{(k)}+s}$

5.2.1. Local Stability of the System with $r_1 \neq 0$ and $r_2 = 0$

In this subsection, $A_{24}^{(k)} = 0$. The characteristic equation of the linearized model (23) is:

$$\Delta^{(k)}(\lambda, \mu_i, r_1) = P^{(k)}(\lambda) + Q^{(k)}(\lambda) e^{-r_1 \lambda} = 0 \tag{24}$$

Where

$$P^{(k)}(\lambda) = (A_{11}^{(k)} - \lambda - \mu_i)[(A_{22}^{(k)} - \lambda - \sigma_2\mu_i)(A_{33}^{(k)} - \lambda - \sigma_3\mu_i) - A_{32}^{(k)} A_{23}^{(k)}] - A_{12}^{(k)} A_{21}^{(k)} (A_{33}^{(k)} - \lambda - \sigma_3\mu_i)$$

And

$$Q^{(k)}(\lambda) = A_{14}^{(k)}[(A_{22}^{(k)} - \lambda - \sigma_2\mu_i)(A_{33}^{(k)} - \lambda - \sigma_3\mu_i) - A_{32}^{(k)} A_{23}^{(k)}]$$

For the equilibria points S_0 and S_2, $A_{14}^{(k)} = 0$, the stability study is realized as in the model with no time delays. Thus, we have the same results like in the instantaneous case. Consequently, the insertion of the time delay r_1 does not have any impact on the stability of these two equilibria points.

Concerning the study of other equilibria points, we use the results relative to the systems stability study in the article [5]. Thus,

(i). $P^{(k)}(\lambda)$ and $Q^{(k)}(\lambda)$ do not have any common imaginary root. Indeed if it was the case, this root should be equal to $\sigma_3\mu_i - A_{33}^{(k)}$, which is impossible.

(ii). $P^{(k)}(0) + Q^{(k)}(0) = (A_{11}^{(k)} - \mu_i + A_{14}^{(k)})[(A_{22}^{(k)} - \sigma_2\mu_i)(A_{33}^{(k)} - \sigma_3\mu_i) - A_{32}^{(k)} A_{23}^{(k)}] - A_{12}^{(k)} A_{21}^{(k)} (A_{33}^{(k)} - \sigma_3\mu_i)$

For each cases, we explain the conditions to obtain $P^{(k)}(0) + Q^{(k)}(0) \neq 0$ to the neighborhood of each equilibrium point.

(iii). It is clear that $\overline{P^{(k)}(-jy)} = P^{(k)}(jy)$ and $\overline{Q^{(k)}(-jy)} = Q^{(k)}(jy)$, with $j^2 = -1$.

(iv). we have $\limsup\{|\frac{Q^{(k)}(\lambda)}{P^{(k)}(\lambda)}| / |\lambda| \to +\infty \text{ and } Re(\lambda) \geq 0\} = 0$. So, $\limsup\{|\frac{Q^{(k)}(\lambda)}{P^{(k)}(\lambda)}| / |\lambda| \to +\infty \text{ and } Re(\lambda) \geq 0\} < 1$.

(v). Let pose $F^{(k)}(y) = |P^{(k)}(jy)|^2 - |Q^{(k)}(jy)|^2$. The function $F^{(k)}$ could be under the following form

$$F^{(k)}(y) = y^6 + \eta_2^{(k)} y^4 + \eta_1^{(k)} y^2 + \eta_0^{(k)} \tag{25}$$

Where:

$$\eta_0^{(k)} = [(A_{11}^{(k)} - \mu_i)(A_{22}^{(k)} - \mu_i\sigma_2)(A_{33}^{(k)} - \mu_i\sigma_3) - A_{32}^{(k)}A_{23}^{(k)}(A_{11}^{(k)} - \mu_i) - A_{12}^{(k)}A_{21}^{(k)}(A_{33}^{(k)} - \mu_i\sigma_3)]^2$$
$$-(A_{14}^{(k)})^2[(A_{22}^{(k)} - \mu_i\sigma_2)(A_{33}^{(k)} - \mu_i\sigma_3) - A_{32}^{(k)}A_{23}^{(k)}]^2,$$

$$\eta_1^{(k)} = -2(A_{11}^{(k)} - \mu_i + A_{22}^{(k)} - \mu_i\sigma_2 + A_{33}^{(k)} - \mu_i\sigma_3)[(A_{11}^{(k)} - \mu_i)(A_{22}^{(k)} - \mu_i\sigma_2)(A_{33}^{(k)} - \mu_i\sigma_3) - A_{32}^{(k)}A_{23}^{(k)}(A_{11}^{(k)} - \mu_i)$$
$$-A_{12}^{(k)}A_{21}^{(k)}(A_{33}^{(k)} - \mu_i\sigma_3)] + [-(A_{11}^{(k)} - \mu_i)(A_{22}^{(k)} - \mu_i\sigma_2 + A_{33}^{(k)} - \mu_i\sigma_3) - (A_{22}^{(k)} - \mu_i\sigma_2)(A_{33}^{(k)} - \mu_i\sigma_3) + A_{32}^{(k)}A_{23}^{(k)}$$
$$+A_{12}^{(k)}A_{21}^{(k)}]^2 - (A_{14}^{(k)})^2[(A_{33}^{(k)} - \mu_i\sigma_3)^2 + (A_{22}^{(k)} - \mu_i\sigma_2)^2 + 2A_{32}^{(k)}A_{23}^{(k)}],$$

$$\eta_2^{(k)} = (A_{11}^{(k)} - \mu_i)^2 + (A_{33}^{(k)} - \mu_i\sigma_3)^2 + (A_{22}^{(k)} - \mu_i\sigma_2)^2 + 2A_{12}^{(k)}A_{21}^{(k)} + 2A_{32}^{(k)}A_{23}^{(k)} - (A_{14}^{(k)})^2$$

The analysis of the local stability of (3) to the neighborhood of S_k is based on the existence of a positive root of $F^{(k)}$. It is apparent that:

$F^{(k)}$ admits a positive root if $\eta_0^{(k)} < 0$.

If $\eta_l^{(k)} > 0, l = 0,1,2$, then, $F^{(k)}$ does not admit any positive root.

Remark 5.2 :

If $\mu_i < \frac{1}{\sigma_3}$ and $\mu_i \neq \frac{A_{22}^{(1)}}{\sigma_2}$ so,

$P^{(1)}(0) + Q^{(1)}(0) \neq 0$ and $F^{(1)}$ admits at least a positive root.

If $\mu_i < \frac{1}{\sigma_3}$ and $\mu_i \neq \frac{A_{22}^{(1)}}{\sigma_2}$ then,

$P^{(3)}(0) + Q^{(3)}(0) \neq 0$ and $F^{(3)}$ allows at least a positive root.

If $\mu_i > \{A_{11}^{(4)}; \frac{A_{22}^{(4)}}{\sigma_2}; \frac{A_{33}^{(4)}}{\sigma_3}\}$ and

$$(A_{11}^{(4)} - \mu_i)(A_{22}^{(4)} - \sigma_2\mu_i) - A_{12}^{(4)}A_{21}^{(4)} < A_{14}^{(4)}(A_{22}^{(4)} - \sigma_2\mu_i)$$

Then, $P^{(4)}(0) + Q^{(4)}(0) \neq 0$ and $F^{(4)}$ admits at least a positive root.

If $\mu_i > \left\{A_{11}^{(5)}; \frac{A_{22}^{(5)}}{\sigma_2}\right\}$ and

$$\left(A_{11}^{(5)} - \mu_i\right)\left(A_{22}^{(5)} - \sigma_2\mu_i\right)\left(A_{33}^{(5)} - \mu_i\right) - A_{32}^{(5)}A_{23}^{(5)}\left(A_{11}^{(5)} - \mu_i\right) - A_{12}^{(5)}A_{21}^{(5)}\left(A_{33}^{(5)} - \sigma_2\mu_i\right) - A_{14}^{(5)}[(A_{11}^{(5)} - \mu_i)(A_{22}^{(5)} - \sigma_2\mu_i) - A_{32}^{(5)}A_{23}^{(5)}] > 0$$

So, $P^{(5)}(0) + Q^{(5)}(0) \neq 0$ and $F^{(5)}$ admits at least a positive root.

Let us find out $\delta(r_1^{(k)}) = sign\{\frac{dRe(\lambda)}{dr_1}|_{\lambda=jy}\}$ the real part sign of a solution λ from the characteristic of the equation $\Delta^{(k)}(\lambda, \mu_i, r_1^{(k)}) = 0$.

Lemma 5.1 : *Let us take account λ as a positive solution of the characteristic of the equation $\Delta^{(k)}(\lambda, \mu_i, r_1) = 0$.*

Let us name $y_1^* = y(r_1^{(k)})$ the positive root of $F^{(k)}$ and $r_1^{(k)}$ the associated time delay verifying, for all $n \in \mathbb{N}$, the following relation: $S_n(r_1^{(k)}) = 0$ with $S_n(r_1^{(k)}) = r_1^{(k)} - \alpha_n(r_1^{(k)})$ and $\alpha_n(r_1^{(k)}) = \frac{1}{y_1^*}\arctan\left(-\frac{Im(\frac{P^{(k)}(jy_1^*)}{Q^{(k)}(jy_1^*)})}{Re(\frac{P^{(k)}(jy_1^*)}{Q^{(k)}(jy_1^*)})}\right) + \frac{2n\pi}{y_1^*}$ if $Re(\frac{P^{(k)}(jy_1^*)}{Q^{(k)}(jy_1^*)}) \neq 0$

Otherwise, $\alpha_n(r_1^{(k)}) = \frac{\pi}{2y_1^*} + \frac{2n\pi}{y_1^*}$.

Let us pose

$$\delta(r_1^{(k)}) = sign\{\frac{dRe(\lambda)}{dr_1}|_{\lambda=jy_1^*}\} = Sgn((F^{(k)})'(y_1^*)Sgn(\frac{dS_n}{dr_1})|_{r_1=r_1^{(k)}}).$$

Whereas $\frac{dS_n}{dr_1} = 1$, so,

$\delta(r_1^{(k)}) = sgn(3(y_1^*)^4 + 2\eta_2^{(k)}(y_1^*)^2 + \eta_1^{(k)})$.

Remark 5.3 : 1. If $\eta_2^{(k)} > 0$ and $\eta_1^{(k)} > 0$ where $(\eta_2^{(k)})^2 - 12\eta_1^{(k)} \leq 0$ so, $\delta(r_1^{(k)}) > 0$.

2. If $(\eta_2^{(k)})^2 - 12\eta_1^{(k)} > 0$, $\eta_1^{(k)} > 0$, $\eta_2^{(k)} < 0$ then,

if $y_1^* \in]0; \sqrt{X_1}[\cup]\sqrt{X_2}; +\infty[$ so, $\delta(r_1^{(k)}) > 0$,

if $y_1^* \in]\sqrt{X_1}; \sqrt{X_2}[$ so, $\delta(r_1^{(k)}) < 0$, with X_1 .and X_2 the solution of the equation $3X^2 + 2\eta_2^{(k)}X + \eta_1^{(k)} = 0$.

3. If $(\eta_2^{(k)})^2 - 12\eta_1^{(k)} > 0$, $\eta_1^{(k)} < 0$ then,

if $y_1^* \in]\sqrt{X_0}; +\infty[$ so, $\delta(r_1^{(k)}) > 0$

if $y_1^* \in]0; \sqrt{X_0}[$ then, $\delta(r_1^{(k)}) < 0$

with X_0 the positive solution of the equation $3X^2 + 2\eta_2^{(k)}X + \eta_1^{(k)} = 0$.

Let us state out the following theorem.

Theorem 5.2 : Let us assume that the stability hypothesis in the instantaneous case are verified. Thus, for $k = 1,3,4,5$

1. there is no change in the stability of the equilibria points S_k in the following case: $\eta_l^{(k)} > 0$, $l = 0,1,2$.

2. We notice the following stability changes:

In the case where $\delta(r_1^{(k)}) > 0$.

If for $r_1 = 0$, S_k was stable, it remains stable when $0 \leq r_1 < r_1^{(k)}$ and becomes unstable if $r_1 > r_1^{(k)}$.

If for $r_1 = 0$, S_k was unstable, it remains for $r_1 \geq 0$.

In the case where $\delta(r_1^{(k)}) < 0$.

If for $r_1 = 0$, S_k was stable, it stays stable when $r_1 \geq 0$.

If for $r_1 = 0$, S_k was unstable, it remains unstable when $0 \leq r_1 < r_1^{(k)}$ and becomes stable if $r_1 > r_1^{(k)}$.

Proof : 1. If $\eta_l > 0, l = 0,1,2$ so, $F^{(k)}$ defined by (25) does not admit any real root.

2. If $\eta_0 < 0$ then, $F^{(k)}$ admits at least a positive root y_1^* associated to the time delay $r_1^{(k)}$. From the reference [5], and by taking into account the remarks (5.2) and (5.3), we have the stability of S_k.■

Remark 5.4 : When $r_2 \neq 0$ and $r_1 = 0$ then, $A_{14}^{(k)} = 0$ and $A_{24}^{(k)} \neq 0$. In that case, by using a similar process to the case $r_1 \neq 0$ and $r_2 = 0$, we study the stability of the equilibria points S_k.

5.2.2. Local Stability of the System with $r_1 \neq 0$ and $r_2 \neq 0$

In this part, the two time delays are considered as non null. We focus on the impact of one of them meanwhile the other one is viewed as a parameter.

The characteristic equation of (23) to the neighborhood of S_k, for all k=0;...;5, is:

$$\Lambda_k(\lambda) = R_1^{(k)}(\lambda) + R_2^{(k)}(\lambda)e^{-r_1\lambda} + \left(R_3^{(k)}(\lambda) + R_4^{(k)}(\lambda)e^{-r_1\lambda}\right)e^{-r_2\lambda} = 0 \quad (26)$$

Where

$$R_1^{(k)}(\lambda) = (A_{11}^{(k)} - \lambda - \mu_i)[(A_{22}^{(k)} - \lambda - \sigma_2\mu_i)(A_{33}^{(k)} - \lambda - \sigma_3\mu_i) - A_{32}^{(k)}A_{23}^{(k)}] - A_{12}^{(k)}A_{21}^{(k)}(A_{33}^{(k)} - \lambda - \sigma_3\mu_i)$$

$$R_2^{(k)}(\lambda) = A_{14}^{(k)}[(A_{22}^{(k)} - \lambda - \sigma_2\mu_i)(A_{33}^{(k)} - \lambda - \sigma_3\mu_i) - A_{32}^{(k)}A_{23}^{(k)}]R_3^{(k)}(\lambda) = A_{24}^{(k)}(A_{11}^{(k)} - \lambda - \mu_i)(A_{33}^{(k)} - \lambda - \sigma_3\mu_i)$$

$$R_4^{(k)}(\lambda) = A_{24}^{(k)}A_{14}^{(k)}(A_{33}^{(k)} - \lambda - \sigma_3\mu_i)$$

When r_1 is the parameter and r_2 the variable, (26) becomes:

$$\Lambda_k(\lambda, \mu_i, r_1, r_2) = P^{(k)}(\lambda) + Q^{(k)}(\lambda)e^{-r_2\lambda} = 0 \quad (27)$$

With $P^{(k)}(\lambda) = R_1^{(k)}(\lambda) + R_2^{(k)}(\lambda)e^{-r_1\lambda}$ and $Q^{(k)}(\lambda) = R_3^{(k)}(\lambda) + R_4^{(k)}(\lambda)e^{-r_1\lambda}$.

However, if r_2 becomes the parameter and r_1 the variable, (26) is:

$$\Lambda_k(\lambda, \mu_i, r_1, r_2) = P^{(k)}(\lambda) + Q^{(k)}(\lambda)e^{-r_1\lambda} = 0 \quad (28)$$

with $P^{(k)}(\lambda) = R_1^{(k)}(\lambda) + R_3^{(k)}(\lambda)e^{-r_2\lambda}$ and $Q^{(k)}(\lambda) = R_2^{(k)}(\lambda) + R_4^{(k)}(\lambda)e^{-r_2\lambda}$. It should be noticed that, for the equilibria points S_k, with $k = 0,1,2,3$, $A_{24}^{(k)} = 0$ or $A_{14}^{(k)} = 0$. So, the stability study to the neighborhood of these equilibria points refers to the previous case .

This is why, we only study the system stability to the neighborhood of the equilibria points S_k for $k = 4,5$.

We are interesting in the impact of the time delay r_2 by keeping r_1 as a parameter. Considering the characteristic of the equation (27) we have:

1. $\overline{P^{(k)}(-jy)} = P^{(k)}(jy)$ and $\overline{Q^{(k)}(-jy)} = Q^{(k)}(jy)$ with $j^2 = -1$.

2. If $|\frac{\mu_i - A_{11}^{(k)}}{A_{14}^{(k)}}| > 1$ so, $P^{(k)}(\lambda)$ and $Q^{(k)}(\lambda)$ have no common imaginary roots.

Indeed, if $P^{(k)}(\lambda)$ and $Q^{(k)}(\lambda)$ have common imaginary roots jy, y would verify the relation $\cos r_1 y = \frac{\mu_i - A_{11}^{(k)}}{A_{14}^{(k)}}$.

Which is impossible because $|\frac{\mu_i - A_{11}^{(k)}}{A_{14}^{(k)}}| > 1$.

3. $\limsup\{|\frac{Q^{(k)}(\lambda)}{P^{(k)}(\lambda)}|/|\lambda| \to +\infty$ and $Re(\lambda) \geq 0\} = 0$. So, $\limsup\{|\frac{Q^{(k)}(\lambda)}{P^{(k)}(\lambda)}|/|\lambda| \to +\infty$ and $Re(\lambda) \geq 0\} < 1$.

4. If $A_{11}^{(k)} - \mu_i < 0$, $A_{22}^{(k)} - \sigma_2\mu_i < 0$ and $A_{33}^{(k)} - \sigma_3\mu_i < 0$ so, $P^{(k)}(0) + Q^{(k)}(0) \neq 0$.

Let us consider the function defined on $\mathbb{R}$ by $F^{(k)}(y) = |P^{(k)}(jy)|^2 - |Q^{(k)}(jy)|^2$. The function $F^{(k)}$ could be under the following form:

$F^{(k)}(y) = G^{(k)}(y) + H^{(k)}(y)$ *with*

$$H^{(k)}(y) = 2H_1^{(k)}(y)\cos(r_1 y) + 2H_2^{(k)}(y)\sin(r_1 y), \quad (29)$$

$$G^{(k)}(y) = y^6 + \eta_2^{(k)}y^4 + \eta_1^{(k)}y^2 + \eta_0^{(k)}, \quad (30)$$

$$H_1^{(k)}(y) = \xi_2^{(k)}y^4 + \xi_1^{(k)}y^2 + \xi_0^{(k)}, \quad (31)$$

$$H_2^{(k)}(y) = \gamma_2^{(k)}y^5 + \gamma_1^{(k)}y^3 + \gamma_0^{(k)}y, \quad (32)$$

Where

$$\eta_2^{(k)} = (E_1^{(k)})^2 + (E_2^{(k)})^2 + (E_3^{(k)})^2 + 2D_1^{(k)} + 2D_2^{(k)} + (D_3^{(k)})^2 - (D_4^{(k)})^2$$

$$\begin{aligned}\eta_1^{(k)} = {} & 2D_1^{(k)}(D_3^{(k)})^2 + 2D_2^{(k)}(E_1^{(k)})^2 + (D_1^{(k)})^2 + (D_2^{(k)})^2 + (E_1^{(k)}E_2^{(k)})^2 + (E_1^{(k)}E_3^{(k)})^2 + (E_2^{(k)}E_3^{(k)})^2 + 2D_1^{(k)}D_2^{(k)} \\ & -2D_1^{(k)}E_1^{(k)}E_2^{(k)} - 2D_2^{(k)}E_2^{(k)}E_3^{(k)} + (D_3^{(k)}E_2^{(k)})^2 + (D_3^{(k)}E_3^{(k)})^2 + 2(D_3^{(k)})^2D_2^{(k)} - (D_4^{(k)}E_1^{(k)})^2 - (D_4^{(k)}E_3^{(k)})^2 \\ & -(D_4^{(k)}D_3^{(k)})^2\end{aligned}$$

$$\eta_0^{(k)} = [E_1^{(k)}(E_2^{(k)}E_3^{(k)} - D_2^{(k)}) - E_3^{(k)}D_1^{(k)}]^2 + [D_3^{(k)}(E_2^{(k)}E_3^{(k)} - D_2^{(k)})]^2 - (D_4^{(k)}E_1^{(k)}E_3^{(k)})^2 - (D_4^{(k)}D_3^{(k)}E_3^{(k)})^2$$

$$\xi_2^{(k)} = D_3^{(k)}E_1^{(k)}$$

$$\xi_1^{(k)} = 2D_2^{(k)}D_3^{(k)}E_1^{(k)} - D_1^{(k)}D_3^{(k)}E_2^{(k)} + D_3^{(k)}E_1^{(k)}(E_2^{(k)})^2 + D_3^{(k)}E_1^{(k)}(E_2^{(k)})^2 - D_3^{(k)}E_1^{(k)}(D_4^{(k)})^2$$

$$\xi_0^{(k)} = D_3^{(k)}E_1^{(k)}(E_2^{(k)}E_3^{(k)})^2 - D_3^{(k)}D_2^{(k)}E_1^{(k)}E_2^{(k)}E_3^{(k)} - D_3^{(k)}D_1^{(k)}E_2^{(k)}(E_3^{(k)})^2 + D_3^{(k)}D_1^{(k)}D_2^{(k)}E_3^{(k)} - D_3^{(k)}D_2^{(k)}E_1^{(k)}E_2^{(k)}E_3^{(k)}$$
$$+D_3^{(k)}(D_1^{(k)})^2E_1^{(k)} - (D_4^{(k)})^2D_3^{(k)}E_1^{(k)}(E_3^{(k)})^2$$

$$\gamma_2^{(k)} = D_3^{(k)}$$

$$\gamma_1^{(k)} = 2D_3^{(k)}D_2^{(k)} + D_3^{(k)}D_1^{(k)} + D_3^{(k)}(E_2^{(k)})^2 + D_3^{(k)}(E_3^{(k)})^2 - (D_4^{(k)})^2D_2^{(k)}$$

$$\gamma_0^{(k)} = D_3^{(k)}D_2^{(k)}D_1^{(k)} - 2D_2^{(k)}D_3^{(k)}E_2^{(k)}E_3^{(k)} + D_3^{(k)}(D_2^{(k)})^2 + D_3^{(k)}(E_3^{(k)}E_2^{(k)})^2 + D_3^{(k)}D_1^{(k)}(E_3^{(k)})^2 - (D_4^{(k)})^2D_3^{(k)}(E_3^{(k)})^2$$

with

$E_1^{(k)} = A_{11}^{(k)} - \mu_i, E_2^{(k)} = A_{22}^{(k)} - \mu_i,\ E_3^{(k)} = A_{33}^{(k)} - \mu_i,\ D_1^{(k)} = A_{12}^{(k)}A_{21}^{(k)},\ D_2^{(k)} = A_{32}^{(k)}A_{23}^{(k)},\ D_3^{(k)} = A_{14}^{(k)}$ and $D_4^{(k)} = A_{24}^{(k)}$.

The following result reveals the existence conditions of a positive root from $F^{(k)}$ to the neighborhood of S_k.
Proposition 5.1 : Let us note

$$Z^{(k)} = \sqrt{(H_1^{(k)})^2 + (H_2^{(k)})^2} \tag{33}$$

$\beta_1 = (\frac{\xi_1^{(k)}}{\xi_2^{(k)}})^2 + \eta_1^{(k)} - \frac{\xi_0^{(k)}}{\xi_2^{(k)}} - \frac{\eta_2^{(k)}\xi_1^{(k)}}{\xi_2^{(k)}}$ and

$$\beta_0 = \frac{\xi_0^{(k)}\xi_1^{(k)}}{(\xi_2^{(k)})^2} + \eta_0^{(k)} - \frac{\eta_0^{(k)}\xi_0^{(k)}}{\xi_2^{(k)}}$$

1. If $\gamma_1^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_1^{(k)}$, $\gamma_0^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_0^{(k)}$ so,
$F^{(k)}$ admits a positive root $\sqrt{\frac{-\beta_0^{(k)}}{\beta_1^{(k)}}}$ if $\beta_0\beta_1 < 0$.
$F^{(k)}$ does not admit any root if not.
2. Let us assume that $|G^{(k)}(y)| < |Z^{(k)}(y)|$.
Let us pose $\forall n \in \mathbb{N}$,

$$\psi_n(y) = y - \frac{1}{r_1}\arctan(\frac{H_2^{(k)}(y)}{H_1^{(k)}(y)}) - \frac{1}{r_1}\arccos(-\frac{G^{(k)}(y)}{Z^{(k)}(y)}) + \frac{2n\pi}{r_1}$$

If n exists such as the equation $\psi_n(y) = 0$ admits a positive solution y_0 so, $F^{(k)}$ admits y_0 like a positive root for any time delay r_1.

3. If $|G^{(k)}(y)| > |Z^{(k)}(y)|$ so, $F^{(k)}$ does not admit any positive roots.

Proof : We have

$$H_1(y) = 0 \Leftrightarrow y^4 = -\frac{1}{\xi_2^{(k)}}(\xi_1^{(k)}y^2 + \xi_0^{(k)})$$

When we replace y^4 by
$-\frac{1}{\xi_2^{(k)}}(\xi_1^{(k)}y^2 + \xi_0^{(k)})$ in $H_2^{(k)}$, we get:

$$H_1^{(k)}(y) = y[(\gamma_1^{(k)} - \frac{\gamma_2^{(k)}\xi_1^{(k)}}{\xi_2^{(k)}})y^2 + \gamma_0^{(k)} - \frac{\gamma_2^{(k)}\xi_0^{(k)}}{\xi_2^{(k)}}.$$

Whereas $\gamma_1^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_1^{(k)}$, $\gamma_0^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_0^{(k)}$ so, $H_2^{(k)}(y) = 0$.

In that case, $F^{(k)}$ is reduced to $G^{(k)}$.

Let us replace again y^4 by $-\frac{1}{\xi_2^{(k)}}(\xi_1^{(k)}y^2 + \xi_0^{(k)})$ in $G^{(k)}$, we have: $G^{(k)}(y) = \beta_1 y^2 + \beta_0$. So, when β_1 and β_0 are all non null and have the same sign then, $G^{(k)}$ does not admit any positive root.

If β_1 and β_0 are opposite signs so, $\sqrt{\frac{-\beta_0^{(k)}}{\beta_1^{(k)}}}$ becomes a unique positive root of $G^{(k)}$, and $F^{(k)}$.

Let us suppose that $|G^{(k)}(y)| < |Z^{(k)}(y)|$ so, $\cos(r_1 y - \varphi_1) = \cos\theta_1$ with $\theta_1 = \arccos(-\frac{G^{(k)}(y)}{Z^{(k)}(y)})$.

Hence $r_1 y = \varphi_1 + \theta_1 + 2n\pi$

Let us pose

$$\psi_n(y) = y - \frac{1}{r_1}\arctan(\frac{H_2^{(k)}(y)}{H_1^{(k)}(y)}) - \frac{1}{r_1}\arccos(-\frac{G^{(k)}(y)}{Z^{(k)}(y)}) + \frac{2n\pi}{r_1}$$

If n exists and the equation $\psi_n(y) = 0$ admits a positive solution y_0 then, $F^{(k)}(y) = 0$ admits y_0 as a positive root for all time delay r_1.

If $\eta_l > 0$ and $|G^{(k)}(y)| > |Z^{(k)}(y)|$ so, $F^{(k)}$ does not admit any positive roots. ■

Remark 5.5 : *If* $A_{11}^{(k)} + A_{22}^{(k)} > 0$, *we have*

$$\gamma_1^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_1^{(k)}, \gamma_0^{(k)}\xi_2^{(k)} = \gamma_2^{(k)}\xi_0^{(k)}$$

$$\Leftrightarrow E_1^{(k)} = E_3^{(k)} = -E_2^{(k)}$$

$$\Leftrightarrow (A_{11}^{(k)} + A_{22}^{(k)})(1-\sigma_3) = (A_{11}^{(k)} - A_{33}^{(k)})(1+\sigma_2).$$

We know that $A_{33}^{(5)} = -p$, $A_{11}^{(5)} > 0$ and $A_{22}^{(5)} > 0$, so, in order to have $(A_{11}^{(5)} + A_{22}^{(5)})(1-\sigma_3) = (A_{11}^{(5)} - A_{33}^{(5)})(1+\sigma_2)$ it is necessary that $1 - \sigma_3 > 0$.

Lemma 5.2: *Let us consider* λ *as a pure imaginary solution of the characteristic of the equation* (27). *Given* $y^* = y(r_2^{(k)})$ *as a positive root of* $F^{(k)}$ *whose time delay is associated to* $r_2^{(k)}$ *; solution of* $r_2 = \frac{\theta(r_2)+2n\pi}{y(r_2)}$ *such as* $cos\theta(r_2) = -Re(\frac{P^{(k)}(jy)}{Q^{(k)}(jy)})$ *and* $sin\theta(r_2) = Im(\frac{P^{(k)}(jy)}{Q^{(k)}(jy)})$.

So, $\lambda_+(r_2^{(k)}) = y^*j$ and $\lambda_-(r_2^{(k)}) = -y^*j$.

We have $r_2^{(k)} = \frac{\theta(r_2^{(k)})+2n\pi}{y(r_2^{(k)})}$

Thus, $r_2^{(k)} = \frac{1}{y^*}\arctan\left(-\frac{Im(\frac{P^{(k)}(jy^*)}{Q^{(k)}(jy^*)})}{Re(\frac{P^{(k)}(jy^*)}{Q^{(k)}(jy^*)})}\right) + \frac{2n\pi}{y^*}$

if $Re(\frac{P^{(k)}(jy^*)}{Q^{(k)}(jy^*)}) \neq 0$.

Otherwise, $r_2^{(k)} = \frac{\pi}{2y^*} + \frac{2n\pi}{y^*}$

$$\delta(r_2^{(k)}) = Sgn\{\frac{dRe(\lambda)}{dr_2}\}_{\lambda=jy^*} = Sgn(F^{(k)})'(y^*)Sgn(\frac{dS_n}{dr_2})/_{r_2=r_2^{(k)}})$$

However, $\frac{dS_n}{dr_2} = 1$, so, if $Sgn(F^{(k)})'(y^*) > 0$ and that $r_2 = 0$ the equilibrium point S_k is stable so, when $r_2 \in [0; r_2^{(k)}]$, it stays stable. If $r_2 > r_2^{(k)}$, then, it becomes stable.

Contrariwise, if it is unstable, it remains.

If $Sgn(F^{(k)})'(y^*) < 0$ and that, $r_2 = 0$, S_5 was stable, it stays. But if it was unstable, it remains until $r_2^{(k)}$. Then, for $r_2 > r_2^{(k)}$, it becomes stable.

If $F^{(k)}$ does not admit any positive roots, there is no stability change.

Proof : See the reference [5].

6. Global Stability of the Instantaneous System

$$C_1 = -1 + \frac{1}{2a} + \frac{M_2}{a^2} + \frac{c}{2a},$$

$$C_2 = -e + \frac{M_3}{d^2} + \frac{1}{2a} + \frac{c}{2a} + \frac{1}{2d} + \frac{pM_3}{2s}$$

and $C_3 = -p + \frac{pM_3}{2s} + \frac{1}{2d}$,

Where M_2 and M_3 are up to set in the global boundedness part of the solutions.

Theorem 6.1 : *Let us assume the theorem 4.1 and the hypothesis* $C_i < 0$ *for* $i = 1,2,3$ *so, the interior equilibrium point* S_5 *is globally and asymptotically stable in* $\mathbb{R}^3$.

Proof : Let us suppose that the theorem 4.1 is checked and the model (14) admits a unique interior equilibrium point $S_5 = (W_1^{**}, W_2^{**}, W_3^{**})$ and it is bounded. Let us pose

Let's pose:

$$l_1(W_1,W_2,W_3)(t,x) = [W_1(t,x) - W_1^{**} - W_1^{**}\ln\frac{W_1(t,x)}{W_1^{**}}] + [W_2(t,x) - W_2^{**} - W_2^{**}\ln\frac{W_2(t,x)}{W_2^{**}}] + W_3^{**}[W_3(t,x) - W_3^{**} - W_3^{**}\ln\frac{W_3(t,x)}{W_3^{**}}] \quad (34)$$

The function l_1 admits zero for the global minimum reached in $(W_1^{**}, W_2^{**}, W_3^{**})$.

Let us pose

$$L_1(W_1,W_2,W_3)(t,x) = \int_\Omega l_1(W_1,W_2,W_3)(t,x)dx \quad (35)$$

Let us demonstrate that the function L as developed is a Lyapunov's function for the system (14).

1. We have : $L_1(W_1^{**}, W_2^{**}, W_3^{**}) = 0$.
2. For any solution (W_1,W_2,W_3) positive of (14), $L_1(W_1,W_2,W_3)$ is positive.
3. Let us prove the following inequality : $\frac{dL_1}{dt} < 0$. We have:

$$\frac{dL_1}{dt} = \int_\Omega \frac{\partial l_1(W_1(t,x),W_2(t,x),W_3(t,x))}{\partial t}dx \quad (36)$$

Therefore,

$$\frac{dL_1}{dt} = \int_\Omega \left[\frac{\partial l_1(W_1,W_2,W_3)(t,x)}{\partial W_1(t,x)}\frac{\partial W_1(t,x)}{\partial t} + \frac{\partial l_1(W_1,W_2,W_3)(t,x)}{\partial W_2(t,x)}\frac{\partial W_2(t,x)}{\partial t} + \frac{\partial l_1(W_1,W_2,W_3)(t,x)}{\partial W_3(t,x)}\frac{\partial W_3(t,x)}{\partial t}\right] dx$$

$$= \int_\Omega \left[\frac{W_1(t,x)-W_1^{**}}{W_1(t,x)}\frac{\partial W_1(t,x)}{\partial t}\right] dx + \int_\Omega \left[\frac{W_2(t,x)-W_2^{**}}{W_2(t,x)}\frac{\partial W_2(t,x)}{\partial t}\right] dx + W_3^{**}\int_\Omega \left[\frac{W_3(t,x)-W_3^{**}}{W_3(t,x)}\frac{\partial W_3(t,x)}{\partial t}\right] dx \tag{37}$$

By using (12), (14) turns into :

$$\begin{cases} \frac{\partial_t W_1(t,x)}{W_1(t,x)} = \frac{\Delta W_1(t,x)}{W_1(t,x)} - (W_1(t,x) - W_1^{**}) \\ -\frac{1}{W_1^{**}+a}(W_2(t,x) - W_2^{**}) \\ +\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x) - W_1^{**}), \\ \\ \frac{\partial_t W_2(t,x)}{W_2(t,x)} = \frac{\sigma_2 \Delta W_2(t,x)}{W_2(t,x)} - e(W_2(t,x) - W_2^{**}) \\ +\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x) - W_1^{**}) \\ +\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x) - W_2^{**}) \\ -\frac{1}{W_2^{**}+a}(W_3(t,x) - W_3^{**}), \\ \frac{\partial_t W_3(t,x)}{W_3(t,x)} = \frac{\sigma_3 \Delta W_3(t,x)}{W_3(t,x)} - \frac{q}{W_2^{**}+s}(W_3(t,x) - W_3^{**}) \\ +\frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}(W_2(t,x) - W_2^{**}) \end{cases} \tag{38}$$

So,

$$\int_\Omega \frac{\partial l_1(W_1,W_2,W_3)}{\partial t} dx = \int_\Omega [W_1(t,x) - W_1^{**}][\frac{\Delta W_1(t,x)}{W_1(t,x)} - (W_1(t,x) - W_1^{**}) - \frac{1}{W_1^{**}+a}(W_2(t,x) - W_2^{**})$$
$$+\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x) - W_1^{**})]dx + \int_\Omega [W_2(t,x) - W_2^{**}][\frac{\sigma_2 \Delta W_2(t,x)}{W_2(t,x)} - e(W_2(t,x) - W_2^{**})$$
$$+\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x) - W_1^{**}) + \frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x) - W_2^{**})$$
$$-\frac{1}{W_2^{**}+d}(W_3(t,x) - W_3^{**})]dx + W_3^{**}\int_\Omega [W_3(t,x) - W_3^{**}][\frac{\sigma_3 \Delta W_3(t,x)}{W_3(t,x)} - \frac{q}{W_2^{**}+s}(W_3(t,x) - W_3^{**})$$
$$+\frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}(W_2(t,x) - W_2^{**})]dx \tag{39}$$

By posing

$$T_1 = \int_\Omega [W_1(t,x) - W_1^{**}][-(W_1(t,x) - W_1^{**}) - \frac{1}{W_1^{**}+a}(W_2(t,x) - W_2^{**}) + \frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)$$
$$-W_1^{**})]dx + \int_\Omega [W_2(t,x) - W_2^{**}][-e(W_2(t,x) - W_2^{**}) + \frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x) - W_1^{**})$$
$$+\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x) - W_2^{**}) - \frac{1}{W_2^{**}+d}(W_3(t,x) - W_3^{**})]dx$$
$$+W_3^{**}\int_\Omega [W_3(t,x) - W_3^{**}][-\frac{q}{W_2^{**}+s}(W_3(t,x) - W_3^{**}) + \frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}(W_2(t,x) - W_2^{**})]dx \tag{40}$$

And

$$T_2 = \int_\Omega [\frac{W_1(t,x)-W_1^{**}}{W_1(t,x)}\Delta W_1(t,x) + \frac{\sigma_2(W_2(t,x)-W_2^{**})}{W_2(t,x)}\Delta W_2(t,x) + W_3^{**}\frac{\sigma_3(W_3(t,x)-W_3^{**})}{W_3(t,x)}\Delta W_3(t,x)]dx \tag{41}$$

We have $T_1 + T_2 = \int_\Omega \frac{\partial l_1(W_1(t,x),W_2(t,x),W_3(t,x))}{\partial t} dx$.

Let us transform T_2. Per Green's formula and considering Neumann's condition ($\frac{\partial W_1}{\partial n} = \frac{\partial W_2}{\partial n} = \frac{\partial W_3}{\partial n} = 0$), we get:

$$T_2 = -W_1^{**} \int_\Omega \frac{|\nabla W_1(t,x)|^2}{W_1(t,x)^2} dx - \sigma_2 W_2^{**} \int_\Omega \frac{|\nabla W_2(t,x)|^2}{W_2(t,x)^2} - \sigma_3 (W_3^{**})^2 \int_\Omega \frac{|\nabla W_3(t,x)|^2}{W_3(t,x)^2} dx \tag{42}$$

Let us compute the supremum value T_1.

$$\begin{aligned} T_1 = \int_\Omega [[-1 + \frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}](W_1(t,x) - W_1^{**})^2 + [-e + \frac{W_3}{(W_2^{**}+d)(W_2(t,x)+d)}](W_2(t,x) - W_2^{**})^2 \\ - \frac{qW_3^{**}}{(W_2^{**}+s)}(W_3(t,x) - W_3^{**})^2 + [-\frac{1}{W_1^{**}+a} + \frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}](W_2(t,x) - W_2^{**})(W_1(t,x) - W_1^{**}) \\ +[-\frac{1}{W_2^{**}+d} + \frac{qW_3^{**}W_3(t,x)}{(W_2^{**}+s)(W_2+s)}](W_2(t,x) - W_2^{**})(W_3(t,x) - W_3^{**})]dx \\ \leq \int_\Omega [[-e + \frac{M_3}{d^2} + \frac{1}{2a} + \frac{c}{2a} + \frac{1}{2d} + \frac{pM_3}{2s}](W_2(t,x) - W_2^{**})^2 + [-1 + \frac{1}{2a} + \frac{M_2}{a^2} + \frac{c}{2a}](W_1(t,x) - W_1^{**})^2 \\ +[-p + \frac{pM_3}{2s} + \frac{1}{2d}](W_3(t,x) - W_3^{**})^2]dx. \end{aligned} \tag{43}$$

So,

$$\begin{aligned} \frac{dL_1}{dt} \leq -\int_\Omega \frac{W_1^{**}}{M_1^2} |\nabla W_1(t,x)|^2 dx - \int_\Omega \sigma_2 (-\frac{W_2^{**}}{M_2^2}) |\nabla W_2(t,x)|^2 dx - \int_\Omega \sigma_3 \frac{W_3^{**}}{M_3} |\nabla W_3(t,x)|^2 dx + \int_\Omega C_1 (W_1(t,x) - W_1^{**})^2 dx \\ + \int_\Omega C_2 (W_2(t,x) - W_2^{**})^2 dx + \int_\Omega C_3 (W_3(t,x) - W_3^{**})^2 dx \end{aligned} \tag{44}$$

So, under the conditions of theorem 6.1, $\frac{dL}{dt} < 0$.

Consequently, the equilibrium point $S_5 = (W_1^{**}; W_2^{**}; W_3^{**})$ of the system is globally and asymptotically stable. ■

7. Global Stability of the System with Time Delays

Theorem 7.1 : Let us assume the hypothesis of the theorems 4.1 and 6.1. Then, r_{01} and r_{02} exist such as, for all $(r_1, r_2) \in [0; r_{01}] \times [0; r_{02}]$, the interior equilibrium point S_5 is globally and asymptotically stable in $\mathbb{R}^3$.

Proof : Let us assume that the theorem 4.1 is verified. Then, the model (3) admits a unique interior point $S_5 = (W_1^{**}, W_2^{**}, W_3^{**})$ and it is bounded. Let us set

$$G(W_1, W_2, W_3)(t,x) = (l_2(W_1, W_2, W_3) + \Sigma)(t,x) \tag{45}$$

With,

$$\begin{aligned} \Sigma(t,x) = \frac{1}{2}\int_{t-r_1}^t \int_y^t \Big|\nabla W_1(s,x)|^2 ds dy + \frac{e\sigma_2}{2}\int_{t-r_2}^t \int_y^t \Big|\nabla W_2(s,x)|^2 ds dy + \frac{r_2 M_2 e^2}{2}\int_{t-r_2}^t (W_2(s,x) - W_2^{**})^2 ds \\ + \frac{M_1}{2}\int_{t-r_1}^t \int_y^t (W_1(s-r_1,x) - W_1^{**})^2) ds dy + \frac{M_2 e^2}{2}\int_{t-r_2}^t \int_y^t (W_2(s-r_2,x) - W_2^{**})^2 ds dy \\ + \frac{M_1}{2a}\int_{t-r_1}^t \int_y^t (W_2(s,x) - W_2^{**})^2 ds dy + \frac{M_1 M_2}{2a^2}\int_{t-r_1}^t \int_y^t (W_1(s,x) - W_1^{**})^2 ds dy \\ + \frac{M_2 ec}{2a}\int_{t-r_2}^t \int_y^t (W_1(s,x) - W_1^{**})^2 ds dy + \frac{M_1 M_3 e}{2d^2}\int_{t-r_2}^t \int_y^t (W_2(s,x) - W_2^{**})^2 ds dy \\ + \frac{M_2 e}{2d}\int_{t-r_2}^t \int_y^t (W_3(s,x) - W_1^{**})^2) ds dy + \frac{r_1 M_1}{2}\int_{t-r_1}^t (W_1(s,x) - W_1^{**})^2 ds \end{aligned} \tag{46}$$

And

$$l_2\big(W_1(t,x),W_2(t,x),W_3(t,x)\big)=\left[W_1(t,x)-W_1^{**}-W_1^{**}\ln\frac{W_1(t,x)}{W_1^{**}}\right]+\left[W_2(t,x)-W_2^{**}-W_2^{**}\ln\frac{W_2(t,x)}{W_2^{**}}\right]$$
$$+W_3^{**}[W_3(t,x)-W_3^{**}-W_3^{**}\ln\tfrac{W_3(t,x)}{W_3^{**}}]. \tag{47}$$

The function l_2 admits zero for the global minimum reached in $(W_1^{**},W_2^{**},W_3^{**})$.
So, $G(W_1,W_2,W_3)(t,x)\geq 0$, with $G(W_1^{**},W_2^{**},W_3^{**})=0$
Let us pose

$$L_2(W_1,W_2,W_3)(t,x)=\int_\Omega G(W_1,W_2,W_3)(t,x)dx \tag{48}$$

Let us show that the function L_2 as developed is a Lyapunov's functional for the system (3).
1. We have : $L_2(W_1^{**},W_2^{**},W_3^{**})=0$
2. For any solution (W_1,W_2,W_3) positive of (3), $L_2(W_1,W_2,W_3)$ is positive.
3. Let us prove the following inequality : $\frac{dL_2}{dt}<0$. We have:

$$\frac{dL_2}{dt}=\int_\Omega\frac{\partial l_2(W_1,W_2,W_3)(t,x)}{\partial t}dx+\int_\Omega\frac{\partial\Sigma(t,x)}{\partial t}dx. \tag{49}$$

One gets:

$$\int_\Omega\partial_t l_2(W_1,W_2,W_3)(t,x)dx$$
$$=\int_\Omega\Big[\frac{\partial l_2(W_1,W_2,W_3)(t,x)}{\partial W_1(t,x)}\frac{\partial W_1(t,x)}{\partial t}+\frac{\partial l_2(W_1,W_2,W_3)(t,x)}{\partial W_2(t,x)}\frac{\partial W_2(t,x)}{\partial t}+\frac{\partial l_2(W_1,W_2,W_3)(t,x)}{\partial W_3(t,x)}\frac{\partial W_3(t,x)}{\partial t}\Big]dx$$
$$=\int_\Omega\frac{W_1(t,x)-W_1^{**}}{W_1(t,x)}\frac{\partial W_1(t,x)}{\partial t}dx+\int_\Omega\frac{W_2(t,x)-W_2^{**}}{W_2(t,x)}\frac{\partial W_2(t,x)}{\partial t}dx+\int_\Omega\frac{W_3(t,x)-W_3^{**}}{W_3(t,x)}\frac{\partial W_3(t,x)}{\partial t}dx. \tag{50}$$

By using (12), (3) becomes:

$$\begin{cases}\frac{\partial_t W_1(t,x)}{W_1(t,x)}=\frac{\Delta W_1(t,x)}{W_1(t,x)}-(W_1(t-r_1,x)-W_1^{**})\\ -\frac{1}{W_1^{**}+a}(W_2(t,x)-W_2^{**})\\ +\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})\\ \frac{\partial_t W_2(t,x)}{W_2(t,x)}=\frac{\sigma_2\Delta W_2(t,x)}{W_2(t,x)}-e(W_2(t-r_2,x)-W_2^{**})\\ +\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})\\ +\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x)-W_2^{**})\\ -\frac{1}{W_2^{**}+a}(W_3(t,x)-W_3^{**})\\ \frac{\partial_t W_3(t,x)}{W_3(t,x)}=\frac{\sigma_3\Delta W_3(t,x)}{W_3(t,x)}-\frac{q}{W_2^{**}+s}(W_3(t,x)-W_3^{**})\\ +\frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}\big(W_{2(t,x)}-W_2^{**}\big).\end{cases} \tag{51}$$

So, by setting

$$T_3=\int_\Omega[W_1(t,x)-W_1^{**}][-(W_1(t-r_1,x)-W_1^{**})-\frac{1}{W_1^{**}+a}(W_2(t,x)-W_2^{**})$$
$$+\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})]dx+\int_\Omega[W_2(t,x)-W_2^{**}][-e(W_2(t-r_2,x)-W_2^{**})$$
$$+\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})+\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x)-W_2^{**})$$

$$-\frac{1}{W_2^{**}+d}(W_3(t,x)-W_3^{**})]dx+W_3^{**}\int_\Omega [W_3(t,x)-W_3^{**}][-\frac{q}{W_2^{**}+s}(W_3(t,x)-W_3^{**})$$

$$+\frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}(W_2(t,x)-W_2^{**})]dx \tag{52}$$

And

$$T_4=\int_\Omega [\frac{W_1(t,x)-W_1^{**}}{W_1(t,x)}\Delta W_1(t,x)+\frac{W_2(t,x)-W_2^{**}}{W_2(t,x)}\sigma_2\Delta W_2(t,x)\ +W_3^{**}\frac{W_3(t,x)-W_3^{**}}{W_3(t,x)}\sigma_3\Delta W_3(t,x)]dx. \tag{53}$$

We have:

$$T_3+T_4=\int_\Omega \partial_t l_2(W_1,W_2,W_3)dx \tag{54}$$

Let us change T_4. Based on Green's formula and taking into account Neumann's condition

$$\frac{\partial W_1}{\partial n}=\frac{\partial W_2}{\partial n}=\frac{\partial W_3}{\partial n}=0 \tag{55}$$

We have :

$$T_4=-W_1^{**}\int_\Omega \frac{|\nabla W_1(t,x)|^2}{W_1(t,x)^2}dx-\sigma_2 W_2^{**}\int_\Omega \frac{|\nabla W_2(t,x)|^2}{W_2(t,x)^2}dx-\sigma_3(W_3^{**})^2\int_\Omega \frac{|\nabla W_3(t,x)|^2}{W_3(t,x)^2}dx. \tag{56}$$

Let us transform T_3.

In view of the relation: for all $i=1,2,\ W_i(t-r_i,x)=W_i(t,x)-\int_{t-r_i}^{t}\frac{\partial W_i(s,x)}{\partial s}ds$,

T_3 changes into:

$$T_3=\int_\Omega [W_1(t,x)-W_1^{**}][-(W_1(t,x)-W_1^{**})-\frac{1}{W_1^{**}+a}(W_2(t,x)-W_2^{**})$$

$$+\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})]dx+\int_\Omega [W_2(t,x)-W_2^{**}][-e(W_2(t,x)-W_2^{**})$$

$$+\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})+\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x)-W_2^{**})$$

$$-\frac{1}{W_2^{**}+d}(W_3(t,x)-W_3^{**})]dx+W_3^{**}\int_\Omega [W_3(t,x)-W_3^{**}][-\frac{q}{W_2^{**}+s}(W_3(t,x)-W_3^{**})$$

$$+\frac{qW_3(t,x)}{(W_2^{**}+s)(W_3(t,x)+s)}(W_2(t,x)-W_2^{**})]dx+\int_\Omega [\int_{t-r_1}^{t}(W_1(t,x)-W_1^{**})\frac{\partial W_1(s,x)}{\partial s}ds$$

$$+\int_{t-r_2}^{t} e(W_2(t,x)-W_2^{**})\frac{\partial W_2(s,x)}{\partial s}ds]dx. \tag{57}$$

Let us pose

$$T_3=T_{31}+T_{32}, \tag{58}$$

where

$$T_{31}=\int_\Omega [W_1(t,x)-W_1^{**}][-(W_1(t,x)-W_1^{**})-\frac{1}{W_1^{**}+a}(W_2(t,x)-W_2^{**})$$

$$+\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})]dx+\int_\Omega [W_2(t,x)-W_2^{**}][-e(W_2(t,x)-W_2^{**})$$

$$+\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}(W_1(t,x)-W_1^{**})+\frac{W_3(t,x)}{(W_2^{**}+d)(W_2(t,x)+d)}(W_2(t,x)-W_2^{**})$$

$$-\frac{1}{W_2^{**}+d}(W_3(t,x)-W_3^{**})]dx+W_3^{**}\int_\Omega [W_3(t,x)-W_3^{**}][-\frac{q}{W_2^{**}+s}(W_3(t,x)-W_3^{**})+\frac{qW_3(t,x)}{(W_2^{**}+s)(W_2(t,x)+s)}(W_2(t,x)-W_2^{**})]dx \tag{59}$$

And

$$T_{32}=\int_\Omega [\int_{t-r_1}^{t}(W_1(t,x)-W_1^{**})\frac{\partial W_1(s,x)}{\partial s}ds+\int_{t-r_2}^{t}e(W_2(t,x)-W_2^{**})\frac{\partial W_2(s,x)}{\partial s}ds]dx \tag{60}$$

Let us compute the supremum value of T_3. For that, first, let us compute the supremum value of T_{31} and T_{32}. Thus,

$$T_{32}=\int_\Omega\int_{t-r_1}^{t}(W_1(t,x)-W_1^{**})\Delta W_1(s,x))dsdx+\sigma_2 e\int_\Omega\int_{t-r_2}^{t}(W_2(t,x)-W_2^{**})\Delta W_2(s,x)dsdx$$

$$+\int_\Omega\int_{t-r_1}^{t}(W_1(t,x)-W_1^{**})W_1(s,x)[-(W_1(s-r_1,x)-W_1^{**})-\frac{W_2(s,x)}{(W_1^{**}+a)(W_1(s,x)+a)}(W_1(s,x)-W_1^{**})$$

$$-\frac{1}{W_1^{**}+a}(W_2(s,x)-W_2^{**})]dsdx+\int_\Omega\int_{t-r_2}^{t}(W_2(t,x)-W_2^{**})W_2(s,x)[-e^2(W_2(s-r_2,x)-W_2^{**})$$

$$-\frac{e}{W_1^{**}+d}(W_3(s,x)-W_3^{**})+\frac{cae}{(W_1^{**}+a)(W_1(s,x)+a)}(W_1(s,x)-W_1^{**})$$

$$+\frac{eW_3(s,x)}{(W_2^{**}+d)(W_2(s,x)+d)}(W_2(s,x)-W_2^{**})]dsdx \tag{61}$$

Finally, we have:

$$T_{32}\le\int_\Omega[\frac{r_1}{2}|\nabla W_1(t,x)|^2+\frac{e\sigma_2 r_2}{2}|\nabla W_2(t,x)|^2+\frac{1}{2}\int_{t-r_1}^{t}|\nabla W_1(s,x)|^2ds+\frac{e\sigma_2}{2}\int_{t-r_2}^{t}|\nabla W_2(s,x)|^2ds]dx$$

$$+\frac{1}{2}\int_\Omega\int_{t-r_1}^{t}W_1(s,x)[[(W_1(t,x)-W_1^{**})^2+(W_1(s-r_1,x)-W_1^{**})^2]+\frac{1}{W_1^{**}+a}[(W_1(t,x)-W_1^{**})^2+(W_2(s,x)-W_2^{**})^2]$$

$$+\frac{W_2(s,x)}{(W_1^{**}+a)(W_1(s,x)+a)}[(W_1(t,x)-W_1^{**})^2+(W_1(s,x)-W_1^{**})^2]]dsdx+\frac{e}{2}\int_\Omega\int_{t-r_2}^{t}W_2(s,x)[e[(W_2(t,x)-W_2^{**})^2$$

$$+(W_2(s-r_2,x)-W_2^{**})^2]+\frac{ca}{(W_1^{**}+a)(W_1(s,x)+a)}[(W_2(t,x)-W_2^{**})^2+(W_1(s,x)-W_1^{**})^2]$$

$$+\frac{W_3(s,x)}{(W_2^{**}+d)(W_2(s,x)+d)}[(W_2(t,x)-W_2^{**})^2+(W_2(s,x)-W_2^{**})^2]$$

$$+\frac{1}{W_2^{**}+d}[(W_2(t,x)-W_2^{**})^2+(W_3(s,x)-W_3^{**})^2]]dsdx \tag{62}$$

So,

$$T_{32}\le\frac{r_1}{2}\int_\Omega|\nabla W_1(t,x)|^2dx+\frac{e\sigma_2 r_2}{2}\int_\Omega|\nabla W_2(t,x)|^2dx+\int_\Omega\frac{r_1M_1}{2}(1+\frac{1}{a}+\frac{M_2}{a^2})(W_1(t,x)-W_1^{**})^2dx$$

$$+\int_\Omega\frac{r_2eM_2}{2}(e+\frac{c}{a}+\frac{M_3}{d^2}+\frac{1}{d})(W_2(t,x)-W_2^{**})^2dx+\frac{1}{2}\int_\Omega\int_{t-r_1}^{t}|\nabla W_1(s,x)|^2dsdx+\frac{e\sigma_2}{2}\int_\Omega\int_{t-r_2}^{t}|\nabla W_2(s,x)|^2dsdx$$

$$+\int_\Omega\int_{t-r_1}^{t}\frac{M_1}{2}(W_1(s-r_1,x)-W_1^{**})^2dsdx+\int_\Omega\int_{t-r_2}^{t}\frac{M_2e^2}{2}(W_2(s-r_2,x)-W_2^{**})^2dsdx$$

$$+\int_\Omega\int_{t-r_1}^{t}\frac{M_1}{2a}(W_2(s,x)-W_2^{**})^2dsdx+\int_\Omega\int_{t-r_1}^{t}\frac{M_1M_2}{2a^2}(W_1(s,x)-W_1^{**})^2dsdx$$

$$+\int_\Omega\int_{t-r_2}^{t}\frac{M_2ec}{2a}(W_1(s,x)-W_1^{**})^2dsdx+\int_\Omega\int_{t-r_2}^{t}\frac{M_2M_3e}{2d^2}(W_2(s,x)-W_2^{**})^2dsdx$$

$$+\int_\Omega\int_{t-r_2}^{t}\frac{M_2e}{2d}(W_3(s,x)-W_3^{**})^2dsdx \tag{63}$$

$$T_{31}=\int_\Omega[[-1+\frac{W_2(t,x)}{(W_1^{**}+a)(W_1(t,x)+a)}](W_1(t,x)-W_1^{**})^2+[-e+\frac{W_3}{(W_2^{**}+d)(W_2(t,x)+d)}](W_2(t,x)-W_2^{**})^2$$

$$-\frac{qW_3^{**}}{(W_2^{**}+s)}(W_3(t,x)-W_3^{**})^2+[-\frac{1}{W_1^{**}+a}+\frac{ca}{(W_1^{**}+a)(W_1(t,x)+a)}](W_2(t,x)-W_2^{**})(W_1(t,x)-W_1^{**})$$

$$+[-\frac{1}{W_2^{**}+d}+\frac{qW_3^{**}W_3(t,x)}{(W_2^{**}+s)(W_2+s)}](W_2(t,x)-W_2^{**})(W_3(t,x)-W_3^{**})]dx$$

$$\leq\int_\Omega[[-1+\frac{1}{2a}+\frac{M_2}{a^2}+\frac{c}{2a}](W_1(t,x)-W_1^{**})^2+[-e+\frac{M_3}{d^2}+\frac{1}{2a}+\frac{c}{2a}+\frac{1}{2d}+\frac{pM_3}{2s}](W_2(t,x)-W_2^{**})^2$$

$$+[-p+\frac{pM_3}{2s}+\frac{1}{2d}](W_3(t,x)-W_3^{**})^2]dx \tag{64}$$

Let us pose $\Gamma=\int_\Omega\frac{\partial l_2(W_1(t,x),W_2(t,x),W_3(t,x))}{\partial t}dx$.

The relations (54) and (58) allow us to conclude that

$$\Gamma\leq\int_\Omega[\left(-\frac{W_1^{**}}{W_1^2}+\frac{r_1}{2}\right)|\nabla W_1(t,x)|^2dx+\sigma_2\left(-\frac{W_2^{**}}{W_2^2}+\frac{er_2}{2}\right)|\nabla W_2(t,x)|^2]dx-\int_\Omega\sigma_3\frac{W_3^{**}}{W_3^2}|\nabla W_3(t,x)|^2dx$$

$$+\int_\Omega(-1+\frac{1}{2a}+\frac{M_2}{a^2}+\frac{c}{2a}+\frac{r_1M_1}{2}+\frac{r_1M_1}{2a}+\frac{r_1M_1M_2}{2a^2})(W_1(t,x)-W_1^{**})^2dx$$

$$+\int_\Omega(-e+\frac{M_3}{d^2}+\frac{1}{2a}+\frac{c}{2a}+\frac{1}{2d}+\frac{pM_3}{2s}+\frac{r_2e^2M_2}{2}+\frac{r_2ecM_2}{2a}+\frac{er_2M_2M_3}{2d^2}+\frac{er_2M_2}{2d})(W_2(t,x)-W_2^{**})^2dx$$

$$+\int_\Omega(-p+\frac{pM_3}{2s}+\frac{1}{2d})(W_3(t,x)-W_3^{**})^2dx+\int_\Omega\int_{t-r_1}^t\frac{1}{2}|\nabla W_1(s,x)|^2dsdx+\int_\Omega\int_{t-r_2}^t\frac{e\sigma_2}{2}|\nabla W_2(s,x)|^2dsdx$$

$$+\int_\Omega\int_{t-r_1}^t\frac{M_1}{2}(W_1(s-r_1,x)-W_1^{**})^2dsdx+\int_\Omega\int_{t-r_2}^t\frac{M_2e^2}{2}(W_2(s-r_2,x)-W_2^{**})^2dsdx$$

$$+\int_\Omega\int_{t-r_1}^t\frac{M_1}{2a}(W_2(s,x)-W_2^{**})^2dsdx+\int_\Omega\int_{t-r_1}^t\frac{M_1M_2}{2a^2}(W_1(s,x)-W_1^{**})^2dsdx+\int_\Omega\int_{t-r_2}^t\frac{M_2ec}{2a}(W_1(s,x)-W_1^{**})^2dsdx$$

$$+\int_\Omega\int_{t-r_2}^t\frac{M_2M_3e}{2d^2}(W_2(s,x)-W_2^{**})^2dsdx+\int_\Omega\int_{t-r_2}^t\frac{M_2e}{2d}(W_3(s,x)-W_3^{**})^2dsdx \tag{65}$$

So, by using the lemma 5.1 of the article [1] $\int_\Omega\frac{\partial\Sigma(t)}{\partial t}dx$ becomes:

$$\int_\Omega\frac{\partial\Sigma(t)}{\partial t}dx=\int_\Omega\frac{1}{2}r_1|\nabla W_1(t,x)|^2dx-\frac{1}{2}\int_\Omega\int_{t-r_1}^t|\nabla W_1(s,x)|^2dsdx+\int_\Omega\frac{r_2e\sigma_2}{2}\left|\nabla W_2(t,x)|^2dx-\frac{e\sigma_2}{2}\int_\Omega\int_{t-r_2}^t\right|\nabla W_2(s,x)|^2dsdx$$

$$+\int_\Omega\frac{r_1M_1}{2}(W_1(t-r_1,x)-W_1^{**})^2)dx-\frac{M_1}{2}\int_\Omega\int_{t-r_1}^t(W_1(s-r_1,x)-W_1^{**})^2)dsdx+\int_\Omega\frac{r_2M_2e^2}{2}(W_2(t-r_2,x)-W_2^{**})^2dx$$

$$-\frac{M_2e^2}{2}\int_\Omega\int_{t-r_2}^t(W_2(s-r_2,x)-W_2^{**})^2dsdx+\int_\Omega\frac{r_1M_1}{2a}(W_2(t,x)-W_2^{**})^2)dx-\frac{M_1}{2a}\int_\Omega\int_{t-r_1}^t(W_2(s,x)-W_2^{**})^2)dsdx$$

$$+\int_\Omega\frac{r_1M_2M_1}{2a^2}(W_1(t,x)-W_1^{**})^2)dx-\frac{M_2M_1}{2a^2}\int_\Omega\int_{t-r_1}^t(W_1(s,x)-W_1^{**})^2)dsdx+\int_\Omega\frac{r_2M_2ec}{2a}(W_1(t,x)-W_1^{**})^2)dx$$

$$-\frac{M_2ec}{2a}\int_\Omega\int_{t-r_1}^t(W_1(s,x)-W_1^{**})^2dsdx+\int_\Omega\frac{r_2M_1M_3e}{2d^2}(W_2(t,x)-W_2^{**})^2dx-\frac{M_1M_3e}{2d^2}\int_\Omega\int_{t-r_2}^t(W_2(s,x)-W_2^{**})^2dsdx$$

$$+\int_\Omega\frac{r_2M_2e}{2d}(W_3(t,x)-W_1^{**})^2)dx-\frac{M_2e}{2d}\int_\Omega\int_{t-r_2}^t(W_3(s,x)-W_1^{**})^2)dsdx+\int_\Omega\frac{r_1M_1}{2}(W_1(t,x)-W_1^{**})^2-\frac{r_1M_1}{2}(W_1(t-r_1,x)$$

$$-W_1^{**})^2dx+\int_\Omega\frac{r_2M_2e^2}{2}(W_2(t,x)-W_2^{**})^2-\frac{r_2M_2e^2}{2}(W_2(t-r_1,x)-W_2^{**})^2dx. \tag{66}$$

To sum up,

$$\frac{dG}{dt} \le \int_\Omega C_4|\nabla W_1(t,x)|^2dx + \int_\Omega C_5|\nabla W_2(t,x)|^2dx + \int_\Omega C_6|\nabla W_3(t,x)|^2dx + \int_\Omega C_7(W_1(t,x) - W_1^{**})^2dx + \int_\Omega C_8(W_2(t,x) - W_2^{**})^2dx + \int_\Omega C_9(W_3(t,x) - W_3^{**})^2dx. \quad (67)$$

Where

$$C_4 = -\frac{W_1^{**}}{M_1^2} + r_1, C_5 = \sigma_2\left(-\frac{W_2^{**}}{M_2^2} + er_2\right), \quad (68)$$

$$C_6 = -\sigma_3\frac{(W_3^{**})^2}{M_3^2}, \quad (69)$$

$$C_7 = C_1 + \frac{r_1M_1}{2a} + r_1M_1 + \frac{r_1M_2M_1}{a^2} + \frac{r_2ecM_2}{2a}, \quad (70)$$

$$C_8 = C_2 + r_2e^2M_2 + \frac{eM_1M_3r_2}{d^2} + \frac{r_2ecM_2}{2a} + \frac{r_2ecM_2}{2d} + \frac{r_1M_1}{2a}, \quad (71)$$

$$C_9 = C_3 + \frac{r_2eM_2}{2d}. \quad (72)$$

Under the hypothesis of the theorem 6.1, $C_i < 0$ for $i = 1,2,3$. Then, r_{01} and r_{02} exist such as, for all $(r_1, r_2) \in [0; r_{01}] \times [0; r_{02}]$, C_i for $i = 4,\ldots,9$ are all inferior to zero. In that case,

$$\frac{dL_2}{dt} < 0 \quad (73)$$

Consequently, the equilibrium $S_5 = (W_1^{**}; W_2^{**}; W_3^{**})$ of the system is globally and asymptotically stable. ■

Remark 7.1: *The global stability analysis shows that the stability established in the model with no time delays remains until the value of* r_1 *and* r_2. *At the neighborhood of these threshold values, there is a stability change. To get these values, it suffices to resolve the system* $C_i < 0$ *for* $i = 4,\ldots,9$. *So, we may conclude that the time delays have a real impact on the stabilities study.*

8. Conclusion

In this paper, we studied a food chain model with diffusion and time delays which implies three species whose corresponding densities are globally bounded. We demonstrated that, these delays inserted in order to heed the internal competition between preys and that of intermediary predators, lead up to a change of the local stability of some equilibria points under certain conditions. We are ending this study with the establishment of the global stability of the interior equilibrium point S_5. So, the delays r_1 and r_2 have a real impact on the global stability of this equilibrium point. Indeed, the stability established in the instantaneous model remains up to a threshold value of delays beyond which a change of global stability is observed. This conclusion remains valid even if, we consider the internal competition between preys for any species' number in presence and in interaction.

Conflicts of Interest

The authors declare that they have no competing interests.

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