The Adomian Decomposition Method of Volterra Integral Equation of Second Kind

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Abstract: In this work, we consider linear and nonlinear Volterra integral equations of the second kind. Here, by converting integral equation of the first kind to a linear equation of the second kind and the ordinary differential equation to integral equation we are going to solve the equation easily. The Adomian decomposition method or shortly (ADM) is used to find a solution to these equations. The Adomian decomposition method converts the Volterra integral equations into determination of computable components. The existence and uniqueness of solutions of linear (or nonlinear) Volterra integral equations of the second kind are expressed by theorems. If an exact solution exists for the problem, then the obtained series convergence very rabidly to that solution. A nonlinear term F(u) in nonlinear volterra integral equations is Lipschitz continuous and has polynomial representation. Finally, the sufficient condition that guarantees a unique solution of Volterra (linear and nonlinear) integral equations with the choice of the initial data is obtained, and the solution is found in series form. Theoretical considerations are being discussed. To illustrate the ability and simplicity of the method. A few examples including linear and nonlinear are provided to show validity and applicability of this approach. The results are taken from the works mentioned in the reference.

1. Introduction

Many problems from physics and other disciplines lead to linear and nonlinear integral equations. Several methods have been proposed for exact solution of these equations "[3]". Adomian "[1, 2]" has presented and developed a so-called decomposition method for solving algebraic, differential, integro- differential, differential-delay, and partial differential equations. The solution is found as infinite series which convergence rapidly to accurate solutions. This method has proven successful in dealing with linear as well as nonlinear problems, as it yields analytical solutions and offers certain advantages over standered numerical solutions. Application of ADM to deferent types of integral equations has discussed by many authors, for examples "[4, 5, 7]", and "[18]". The integral equation originates from the conversion of a boundary-value problem or an initial-value problem, but many problems lead directly to integral equation and cannot

be formulated in terms of the problem easier, or sometimes enable us, to prove fundamental results on the existence and uniqueness of the solution. An integral equation is an equation in which the unknown function u(x) to be determined appears under the integral sign "[7]"and "[13]". A typical form of an integral equation in u(x) is of the form

$$\varphi(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \qquad (1)$$

where $k(x,t)$ is called the kernel of integral equation, $\lambda \neq 0$ the parameter of integral equation, $\alpha(x)$ and $\beta(x)$ are the limits of integration. It is to be noted here that both the kernel $k(x,t)$ and the function $f(x)$ in equation are given function. The kernel is always defined and continuous on $D = \{(x,t): \alpha(x) \leq x \leq \beta(x), \alpha(x) \leq t \leq \beta(x)\}$.

When $\varphi(x) = 0$, the equation is of the first kind; otherwise, it is of the second kind. An integral equation can be classified as a linear or nonlinear integral equation as we

know in the ordinary and partial differential equations "[10, 12, 13]". The most frequently used integral equations fall under two major classes, namely Volterra and Fredholm integral equations.

Definition

Volterra integral equations are written in a form (1) where the upper limit of integration $\beta(x) = x$ (independent variable). The most standard form of Volterra linear integral equations are of the form

$$\varphi(x)u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt$$

2. Significance of the Study

Integral equations are often easier to solve, more elegant and compact that a corresponding differential equation. Because it does not require supplementary initial or boundary conditions and the contribution of this study is that:

(1) Distinguish the importance of integral equations over the differential equation.
(2) Initiate other researchers for depth study about integral equations different numerical techniques.
(3) It gives clue to extend the concept of integral equation to many interdisciplinary areas instead of using differential equations.
(4) It invites other researchers interested on the rest part of integral equation.

3. Relationship between Integral Equations and Differential Equations

The theories of ordinary and partial differential equations are fruitful source of integral equations. The researcher shall sketch here one of the ways in which integral equations can arise from ordinary differential equations. Most ordinary differential equations can be expressed as integral equations, but the reverse is not true "[14]" and"[18]".To investigate the relationship between integral and differential equations, the researcher will need the following lemma which will allow us to replace a double integral by a single one.

Lemma1. (Replacement lemma):

Suppose that u: $[a, b] \rightarrow \mathbb{R}$ is continuous. Then

$$\int_a^x \int_a^{x_1} u(t)dtdx_1 = \int_a^x (x-t)u(t)dt, x \in [a, b]$$

Now considering the first order differential equation.

$$\frac{\partial y}{\partial x} = y'(x) = u(x,y)$$ (2)

with the initial condition $y(0) = y_0$ if we say, u(x,y) is continuous function of (x,y), integrate (2) from 0 to x, obtaining

$$\int_0^x \left(\frac{dy}{dx}\right) dx = \int_0^x u(x,y) dx$$

$$y(x) = y_0 + \int_0^x u(t,y(t))dt$$

This illustrates the general fact that, by going over to integral equations, it includes both the differential equation and the initial conditions in a single equation. Again consider the second order differential equation.

$$\frac{d^2y}{dx^2} = y''(x) = u(x,y),$$ (3)

With initial conditions $y(0) = y_0, y'(0) = y_1$. Then integrate (3) from 0 to x, obtaining

$$y'(x) = y_1 + \int_0^x u(t,y(t))dt$$

Whence the second integration

$$y(x) = y_0 + y_1 x + \int_0^t (x-t) u(t,y(t))dt$$ (4)

The argument is reversible; so that here again the differential equation (3), together with the initial conditions, is equivalent to the single integral equation (4).We see also that any solution of (3) satisfies an integral equation of the form.

$$y(x) = A + Bx + \int_0^x (x-t)u(t,y(t))dt$$ (5)

The constants A and B being determined by the initial conditions. They may also be determined in other ways. Suppose, for instance, that y(x) is required to satisfy a two point boundary condition, say $y(0) = \alpha, y(l) = \beta$, if (x = l) Substituting in (5), we obtain.

$$\alpha = y(0) = A,$$

$$B = \frac{\beta - \alpha}{l} - \frac{1}{l} \int_0^l (l-t) u(t,y(t)) dt$$

Hence, the function y(x) must therefore satisfy the integral equation

$$y(x) = \alpha + \frac{\beta - \alpha}{l} x$$

$$+ \int_0^x (x-t) u(t,y(t)) dt$$

$$- \frac{x}{l} \int_0^l (l-t) u(t,y(t))dt$$

Which we can be written in the form

$$y(x) = F(x) - \int_0^l k(x,t)\, u\big(t, y(t)\big)\, dt \qquad (6)$$

where $F(x) = \alpha + \frac{\beta - \alpha}{l} x$

$$k(x,t) = \begin{cases} \dfrac{t(l-x)}{l}, & for\ 0 \le t \le x \\[2mm] \dfrac{x(l-t)}{l}, & for\ x \le t \le l \end{cases}$$

Here k(x, t) is the kernel of the equation and the argument is again reversible, so that (6) is equivalent to (3) together with the boundary conditions. If the differential equation is linear, we are led in this way to a linear integral equation of the second kind.

4. Solution of Linear Volterra Integral Equation

This section deals with the Volterra integral equations and their solution techniques. The solution of this integral equation is the unknown function u(x) which satisfies that equation, but there is the problem that the upper limit of the integral is the independent variable of the equation. For thus choose a quadrature scheme that utilizes the endpoints of the interval; otherwise we will not be able to evaluate the functional equation at the relevant quadrature points. There are a host of solution techniques to deal with the Volterra integral equations.

4.1. Existence and Uniqueness of Solutions of Linear Volterra Integral Equations of the Second Kind

Let $\mathcal{A}$ be an arbitrary nonempty closed subset of the real number, for $t \in \mathcal{A}$ we define the operator $\sigma : \mathcal{A} \to \mathcal{A}$ as follows:

$$\sigma(t) = inf\ \{s \in \mathcal{A} : s > t\}$$

We note that $\sigma(t) \ge t$ for any $t \in \mathcal{A}$, for $t \in \mathcal{A}$ we define the operator $\rho : \mathcal{A} \to \mathcal{A}$ by

$$\rho(t) = sup\ \{s \in \mathcal{A} : s < t\}$$

We note that $\rho(t) \le t$ for any $t \in \mathcal{A}$. Now we say that t is right dense if $t < sup\ \mathcal{A}$ and $\sigma(t) = t$. we say that t left dense if $t > inf\ \mathcal{A}$ and $\rho(t) = t$.

Definition:

A function $f : \mathcal{A} \to \mathbb{R}$ is called rd-continuous provide it is continuous at right dense points in $\mathcal{A}$ and its left sided limits exist (finite) at left dense points in $\mathcal{A}$.

Now we can state the following theorems.

Theorem1. (Existences of Solutions of Volterra Integral Eq.s of Second kind):

Let $k(x,t)$ be a real-valued rd continuous function defined on $a \le x, t \le b, f(x)$ be a real valued rd-continuous function defined on $a \le x \le b$. Let also,

$|f(x)| \le M$ for all $x \in [a, b]$,

$|k(x,t)| \le N\ for\ all\ x, t \in [a, b]$.

Then for $|\lambda| < \infty$ the Volterra integral equation of the

second kind

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)\, dt, a \le x \le b \qquad (7)$$

has a rd-continuous solution u(x) defined on [a, b].

Theorem2. (Uniqueness of Solutions of Volterra Integral Eq.s of Second Kind):

Suppose that all conditions of Theorem 1 are fulfilled. Then the Eq. (7) has unique rd-continuous solution u(x) defined on [a, b].

4.2. The Method of Adomian Decompositions for Linear Volterra Integral Equation

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any Volterra integral equation into a sum of an infinite number of components by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (8)$$

where the components $u_n(x), n \ge 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_0(x), u_1(x), u_2(x), \ldots$ individually. The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (8) into the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \qquad (9)$$

we obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t)\big(\sum_{n=0}^{\infty} u_n(t) \big)dt \quad (10)$$

with u_0 identified as all terms out of the integral sign. Consequently, the components $u_n(t), n \ge 1$ of the unknown function $u(x)$ are completely determined in a recurrent manner if we set

$$u_0(x) = f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t)\, dt, n \ge 0 \qquad (11)$$

that is equivalent to

$$u_0(x) = f(x),$$

$$u_1(x) = \lambda \int_0^x K(x,t)u_0(t)\, dt,$$

$$u_2(x) = \lambda \int_0^x k(x,t)u_1(t)dt \qquad (12)$$

$$u_3(x) = \lambda \int_0^x K(x,t)u_2(t)\, dt$$

and so on for other components. In view of (12), the components $u_0(x), u_1(x), u_2(x), \ldots$ are completely determined. As a result, the solution u(x) of (9) in a series

form is obtained by using the series (8). In the other words, the Adomian decomposition method converts the Volterra integral equation into a determin- -ation of computable components. It was formally shown by many researchers that if an exact solution exists for the problem, then the obtained series converge very rapidly to that solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series.

However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms are usually used for numerical purposes. The more components we use the higher accuracy we obtain.

In recent decades, there has been a great deal of interest in the Adomian decomposition method. Recently, a modification of the Adomian decomposition method was proposed in "[11]". However, the modified method was established based on the assumption that the function f can be divided into two parts, and its success depends mainly on the proper choice of the parts f_1 and f_2 .The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. Therefore it cannot be used if the function $f(x)$ consists of only one term. To give a clear description of the technique, we can set

$$f(x) = f_1(x) + f_2(x) \qquad (13)$$

In view of (13), we introduce a qualitative change in the formation of the recurrence relation (11). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$$u_0(x) = f_1(x)$$

$$u_1(x) = f_2(x) + \lambda \int_0^x k(x,t)u_0(t)dt \qquad (14)$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)u_n(t) \, dt \ n \geq 1$$

Note that, the proper selection of the functions $f_1(x)$ and $f_2(x)$, may be obtained the exact solution $u(x)$ by using very few iterations, and sometimes by evaluating only two components.

4.3. Numerical Examples of Linear Volteera Integral Equations

This section contains some examples of linear Volterra integral equations.
Example 4.3.1
Consider the following linear Volterra integral equation

$$u(x) = x + \int_0^x (t-x)u(t)dt \qquad (15)$$

Recall that the solution $u(x)$ is assumed to have a series

form given in (8). Substituting the decomposition series (8) into both sides of (15) gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (t-x) \sum_{n=0}^{\infty} u_n(t) \, dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= x$$
$$+ \int_0^x (t-x)[u_0(x) + u_1(x) + u_2(x)$$
$$+ \cdots]dt$$

we obtain the following recurrence relation:

$$u_0 = f(x),$$

$$u_{n+1}(x) = \int_0^x (t-x)u_n(t) \, dt, n \geq 0$$

that gives $u_0 = x$

$$u_1 = \int_0^x (t-x)u_0(t) \, dt$$

$$u_1(x) = \int_0^x (t-x)t \, dt = -\frac{x^3}{3!}$$

$$u_2(x) = \int_0^x (t-x)u_1(t) \, dt$$

$$= \int_0^x (t-x)\left(-\frac{t^3}{3!}\right) dt = \frac{x^5}{5!}$$

$$u_3(x) = \int_0^x (t-x)u_2(t) \, dt - \frac{x^7}{7!}$$

$$= \int_0^x (t-x)\left(\frac{t^5}{5!}\right) dt = -\frac{x^7}{7!}$$

and so on. Using (8) gives the series solution:

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

that converges to the closed form solution:

$$u(x) = sin \, x$$

Example 4.3.2
Consider the following linear Volterra integral equation of the second kind:

$$u(x) = \cos x + \sin x - \int_0^x u(t)dt$$

We first split $f(x)$ given by

$$f(x) = \cos x + \sin x$$

into two parts, namely

$$f_1(x) = \cos x \,, \, f_2(x) = \sin x$$

We next use the modified recurrence

$$u_0(x) = f_1(x) = \cos x$$

$$u_1(x) = f_2(x) + \lambda \int_0^x K(x,t)u_0(x)\,dt$$

$$= \sin x - \int_0^x u_0(x)\,dt = 0,$$

$$u_{n+1}(x) = -\int_0^x K(x,t)u_n(x)\,dt = 0, n \geq 1$$

It is obvious that each component of $u_n(x), n \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = \cos x$$

5. Solution of Nonlinear Volterra Integral Equation

In this section we will study the existence and uniqueness of the general nonlinear Volterra integral equation's which to be in the form

$$u(x) = f(x) + \int_0^x F\big(x,t;u(t)\big)dt \qquad (16)$$

where F and f are L_2 −functions.

Suppose that for any pair z_1, z_2 we can write

$$|F(x,t; z_1) - F(x,t; z_2)| \leq \alpha(x,t)|Z_1 - Z_2| \quad (17)$$

and further, that we have

$$\left|\int_0^x F\big(x,t;u(t)\big)dt\right| \leq b(x) \qquad (18)$$

Where $\alpha(x,t)$ and $b(x)$ are any two L_2 −functions such that in the entire basic domain $(0 \leq t \leq x \leq h)$ we have

$$\int_0^h dx \int_0^x \alpha^2(x,t)dt \leq A^2$$

$$\int_0^x b^2(t)dt \leq B^2 \qquad (19)$$

Where A^2 and B^2 are two positive constants.

As in the linear case we found the solution of our equation as the limit of a sequence $\{u_n(x)\}$ of functions whose first

approximation element is the given function $u_0(x) \equiv f(x)$; the other approximations are evaluated by the iteration formula

$$u_n(x) = f(x) + \int_0^x F(x,t; u_{n-1}(t))dt, (n = 1,2, \dots) \quad (20)$$

Theorem 3:

Suppose that f and F are L_2 − functions satisfying conditions (17), (18) and (19), then the nonlinear Volterra integral equation (16) have a unique solution $u(x)$ of the class L_2 − functions. This solution is series of the form $u(x) = \sum_{n=0}^\infty u_n(x)$ which converges absolutely and almost uniformly.

5.1. The Method of Adomian Decomposition for Nonlinear Volterra Integral Equation

In the nonlinear Volterra integral equation,

$$u(x) = f(x) + \int_0^x k(x,t)F\big(u(t)\big)dt \qquad (21)$$

the nonlinear term $F(u)$ is Lipschitz continuous with $|F(u) - F(z)| \leq L\,|u - z|$ and has the Adomian polynomial representation

$$F(u) = \sum_{n=0}^\infty A_n(u_0, u_1, u_2, \dots, u_n) \qquad (22)$$

Where the traditional formula for A_n is

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)\left[F\left(\sum_{i=0}^n \lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \dots \quad (23)$$

There is another programmable formula for the Adomian polynomial

$$A_n = F(S_n) - \sum_{i=0}^{n-1} A_i \qquad (24)$$

Where the partial sum is

$$S_n = \sum_{j=0}^n u_j(x).$$

Application of ADM to (21) yields

$$u(x) = \sum_{i=0}^\infty u_i(x) \qquad (25)$$

Where

$$u_0(x) = f(x)$$

$$u_i(x) = \int_0^x k(x,t)A_{i-1}\,dt, i \geq 1.$$

Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1 , A_2 depends only on $u_0, u_1, and\ u_2$, and so on. Second, substituting (23) into (22) gives

$$F(u) = A_0 + A_1 + A_2 + A_3 + \cdots$$
$$= F(u_0) + (u - u_0)F'(u_0)$$
$$+ \frac{1}{2!}(u - u_0)^2 F''(u_0) + \cdots$$

5.2. Numerical Examples of Nonlinear Volteera Integral Equations

In this section we introduce some examples.
Example5.2.1:
Consider the nonlinear Volterra integral equation

$$u(x) = x + \int_0^x u^2(t)\, dt, \qquad (26)$$

The Adomian polynomials can be generated using formula (23) or formula (24). Formula (24) is programmable and the Adomian series solution can be converged faster when using it. The first four polynomials using formulas (23) are computed to be:

$$A_0 = u_0^2$$
$$A_1 = 2u_0 u_1$$
$$A_2 = u_1^2 + 2u_0 u_2$$
$$A_3 = 2u_1 u_2 + 2u_0 u_3$$
$$A_4 = u_2^2 + 2u_1 u_3 + 2u_0 u_4$$

Substituting the series (25) and the Adomian polynomials (22) into the left side and the right side of (25) respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x \sum_{n=0}^{\infty} A_n(t)\, dt$$

We set

$$u_0(x) = x, \quad u_{n+1}(x) = \int_0^x A_n(t)\, dt, n \geq 0$$

This in turn gives

$$u_0(x) = x, \quad u_1(x) = \int_0^x u_0^2(t)\, dt = \frac{1}{3}x^3,$$

$$u_2(x) = \frac{2}{15}x^5, \ldots$$

and so on. The solution in a series form is given by

$$u(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

that converges to the exact solution
$$u(x) = \tan x$$

6. Conclusion.

In this paper we proposed of the some numerical methods to solve Volterra (linear and nonlinear) integral equations.

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)\, dt, a \leq x \leq b$$

And

$$u(x) = f(x) + \lambda \int_0^x k(x,t)F(u(t))\, dt, a \leq x \leq b$$

To solve these for u(x) we set

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

plays an essential role on the speed of the convergence of the Numerical methods for solving Volterra integral equation. The sufficient condition that guarantees a unique solution to the given problem is obtained.

References

[1] G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, San Diego, CA, 1986.

[2] G. Adomian, Solution of physical problems by decomposition, Appl. Math. Comput. 27 (9/10) (1994) 145-154.

[3] K. E. Atkinson, A survey of numerical methods for solving nonlinear integral equations. Integral Equations Appl., 1992, 4, 15–40.

[4] J. Biazar, E. Babolian, Solution of a system of nonlinear Volterra integral equations of the second kind, Far East J. Math. Sci. 2 (6) (2000)935–945.

[5] E. Babolian, J. Biazar, A. R. Vahidi, The decomposition method applied to systems of Fredholm integral equations of the second kind, Appl. Math. Comput. 148 (2004) 443–452.

[6] I. L. El-Kalla, Convergence of the Adomian method applied to a class of nonlinear integral equations, Elsevier Ltd. All rights reserved, 21 (2008) 372–376.

[7] H. Sadeghi Goghary, Sh. Javadi, E. Babolian, Restarted Adomian method for system of nonlinear Volterra integral equations, Appl. Math. Comput. 161 (2005) 745–751.

[8] H. Jafari, E. Tayyebi1, S. Sadeghi, C. M. Khalique, A new modification of the Adomian decomposition method for nonlinear integral equations, Int. J. Adv. Appl. Math, and Mech. 1(4) (2014) 33 – 39.

[9] M. Rahman. (2007). Integral Equations and their Applications, 1st Ed; WIT Press.

[10] P. Linz. (1985). Analytical and Numerical Methods for Volterra Equations.

[11] Porter, David. (1990). Integral equations A practical treatment, from spectral theory to applications, first edition.

[12] F. Smithies (1958). Cambridge Tracts in Mathematics and Mathematical Physics.

[13] Peter J. Collins. (2006). Differential and Integral equations, WIT Press.

[14] T. A. Burton Eds. (2005). Volterra Integral and Differential Equations, 2nd ed; Academic Press, Elsevier.

[15] S. Abbasbandy, Numerical solution of the integral equations: Homotopy perturbation method and Adomian decomposition method, Appl. Math. Comput. 173 (2006) 393–500.

[16] Stephen M. Zemyan, the Classical Theory of Integral Equations, Springer, Pennsylvania State University Mont Alto Mont Alto, PA, USA, 2010.

[17] Svetlin G. Georgiev, Integral Equations on Time Scales, Atlantis Press and the author(s) 2016.

[18] A. Wazwaz. A First Course in Integral Equations -Solutions Manual, 2nd ed; World Scientific Publishing Co. Pte. Ltd.

[19] F. G. Trkloml, Integral Equations First published (1957) 49, Library of congress catalog number 57-1.