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On the Planarity of G^{++-}

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Abstract: Let G be a simple graph. The transformation graph G^{++-} of G is the graph with vertex set $V(G) \bigcup E(G)$ in which the vertex x and y are joined by an edge if and only if the following condition holds:(i) $x, y \in V(G)$ and x and y are adjacent in G, (ii) $x, y \in E(G)$, and x and y are adjacent in G, (iii) one of x and y is in $V(G)$ and the other is in $E(G)$, and they are not incident in G. In this paper, it is shown G^{++-} is planar if and only if $|E(G)| \le 2$ or G is isomorphic to one of the following graphs: C_3, $C_3 + K_1$, P_4, $P_4 + K_1$, $P_3 + K_2$, $P_3 + K_2 + K_1$, $K_{1,3}$, $K_{1,3} + K_1$, $3K_2$, $3K_2 + K_1$, $3K_2 + 2K_1$, C_4, $C_4 + K_1$.

Keywords: Total Graph, Planarity, Transformation Graph

1. Introduction

All graphs considered here are finite, simple and undirected. Undefined terminology and notation can be found in [2]. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ is called the order of G. $|E(G)|$ is called the size of G. The neighborhood $N_G(v)$ of v is the set of all vertices of G adjacent to v. Since G is simple, $|N_G(v)| = d_G(v)$.

Suppose that V' is a nonempty subset of $V(G)$. The subgraph $G[V']$ of G induced by V' is a graph with $V(G[V']) = V'$ and $uv \in E(G[V'])$ if and only if $uv \in E(G)$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The union $G \bigcup H$ of G and H is the graph whose vertex set is $V(G) \bigcup V(H)$ and the edge set $E(G) \bigcup E(H)$. Particularly, we denote their union by $G + H$ if they are disjoint, i.e. $V(G) \cap V(H) = \emptyset$.

The line graph $L(G)$ of G is the graph whose vertex set is $E(G)$, and in which two vertices are adjacent if and only if they are adjacent in G. The total graph G^{+++} of G is the graph whose vertex set is $V(G) \bigcup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in G. Wu and Meng [9] generalized the concept of total graph, and introduced some new graphical transformations. We adopt the symbol G^{xyz} with

$x, y, z \in \{+, -\}$ introduced in [9].

A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their end vertices. A subdivision of a graph G is a graph that can be obtained from G by a sequence of edges subdivisions. Behzad [1] characterized the graphs G for which G^{+++} is planar. Liu [8] give a necessary and sufficient condition for a graph G for which G^{---} is planar. Wu et al. [10] proved that G^{-++} is planar if and only if the order of G is at most 4. We refer to [4, 5, 6, 7, 10, 12, 13] for more relevant results on G^{xyz}. As usual, the complete graph, the cycle, the path of order n are denoted K_n, C_n, P_n, respectively.

We use the well-known theorem of Kuratowski [2] in Section 2.

Theorem 1.1. *A graph is planar if and only if it contains no subdivision of* K_5 *or* $K_{3,3}$.

Corollary 1.2. *Every simple planar graph has a vertex of degree at most five.*

Our main result is given as follows.

Theorem 1.3. *Let G be a graph of size m. Then G^{++-} is planar if and only if either $m \le 2$ or G is isomorphic to one of the following graphs:* C_3, $C_3 + K_1$, P_4, $P_4 + K_1$, $P_3 + K_2$, $P_3 + K_2 + K_1$, $K_{1,3}$, $K_{1,3} + K_1$, $3K_2$, $3K_2 + K_1$, $3K_2 + 2K_1$, C_4, $C_4 + K_1$

Proof. It is immediate form the results of Lemmas 2.1-2.5.

2. Proof

We start with a trivial observation.

Lemma 2.1. If H is a subgraph of G, then H^{++-} is a subgraph of G^{++-}.

In particular, by Lemma 2.1, if H^{++-} is nonplanar and $G = H + kK_1$ for an integer $k \geq 1$, then G^{++-} is nonplanar. One can easily check that G^{++-} is planar for each G of size $m \leq 2$. Next we consider the graphs of size 3. There are precisely five graphs of size 3 without isolated vertex as shown in Figure 1.

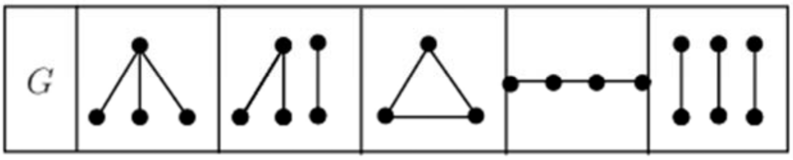

Figure 1. *All graphs of size 3 with no isolated vertices.*

Lemma 2.2. *For a graph G of size 3, G^{++-} is planar if and only if* $G \in \{C_3, C_3 + K_1, P_4, P_4 + K_1, P_3 + K_2, P_3 + K_2 + K_1, K_{1,3}, K_{1,3} + K_1, 3K_2, 3K_2 + K_1, 3K_2 + 2K_1\}$.

Proof. The sufficiency. As illustration in Figure 2, the transformation graphs G^{++-} of $C_3 + K_1$, $P_4 + K_1$, $P_3 + K_2 + K_1$, $K_{1,3} + K_{1,}3K_2 + 2K_1$ are planar. By Lemma 2.1, the transformation graphs G^{++-} of C_3, P_4, $P_3 + K_2$, $K_{1,3}$, $3K_2$, $3K_2 + K_1$ are planar.

The necessity. For each $G \in \{C_3 + 2K_1, P_4 + 2K_1, P_3 + K_2 + 2K_1, K_{1,3} + 2K_1, 3K_2 + 3K_1\}$ the transformation graph $(G + 2K_1)^{++-}$ of G is nonplanar since it contain a subdivision of K_5 or $K_{3,3}$, as shown in Figure 3.

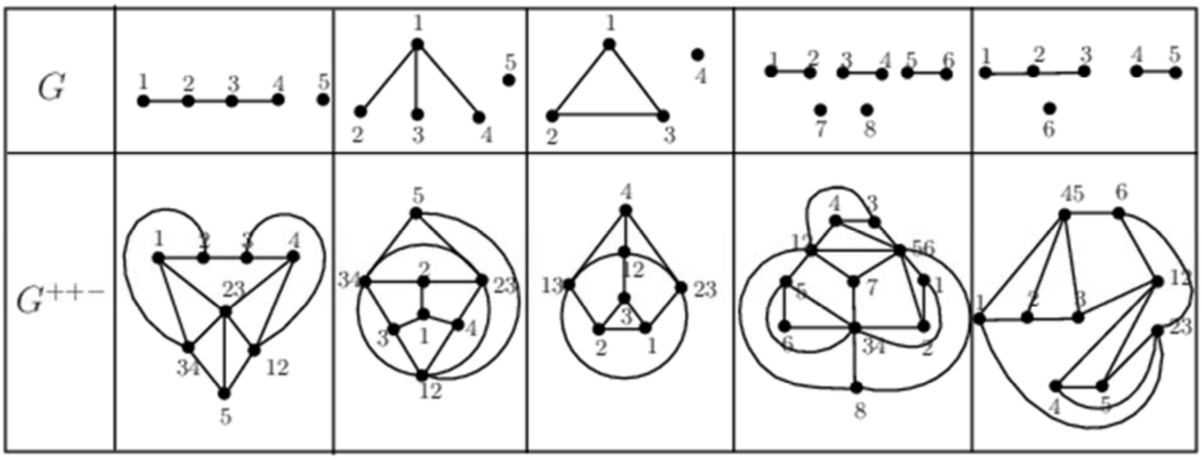

Figure 2. *Transformation graphs G^{++-} of $C_3 + K_1$, $P_4 + K_1$, $P_3 + K_2 + K_1$, $K_{1,3} + K_1$, $3K_2 + 2K_1$.*

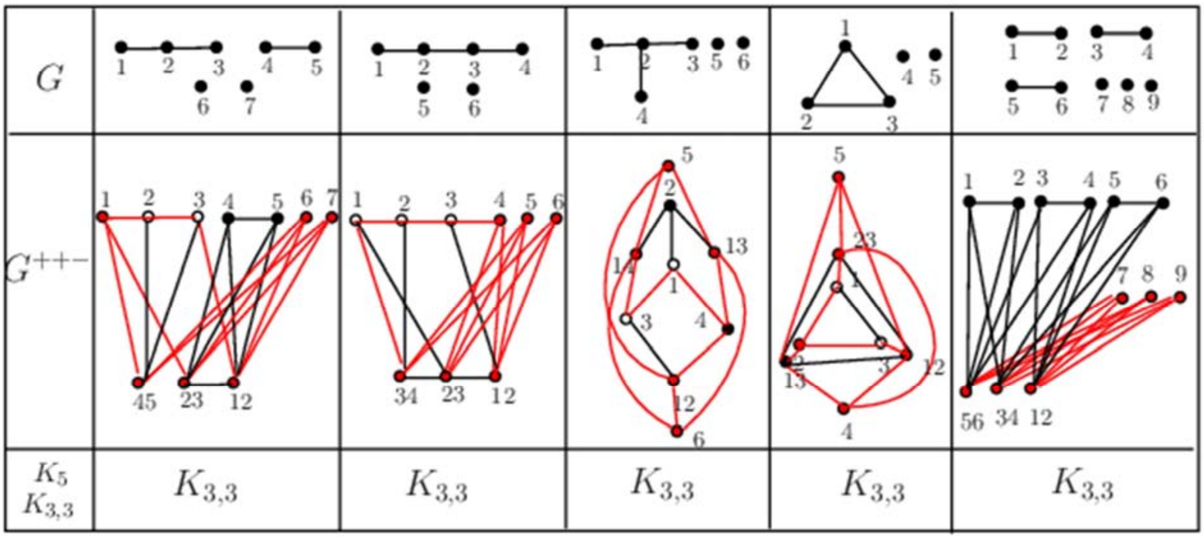

Figure 3. *Transformation graphs G^{++-} of $C_3 + 2K_1$, $P_4 + 2K_1$, $P_3 + K_2 + 2K_1$, $K_{1,3} + 2K_1$, $3K_2 + 3K_1$.*

Now we consider the graphs of size 4. There are precisely eleven graphs of size 4 without isolated vertex as shown in Figure 4.

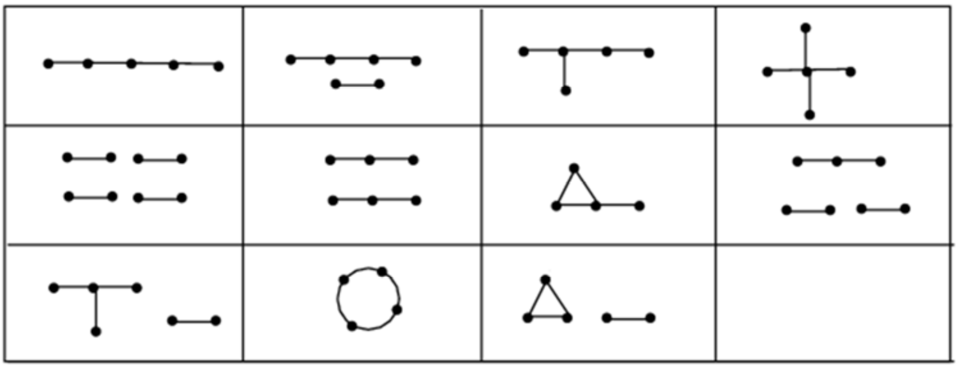

Figure 4. *All graphs of size 4 with no isolated vertices.*

Lemma 2.3. *For a graph G of size 4, G^{++-} is planar if and only if* G∈ {C_4, C_4 +K_1}.

Proof. The sufficiency. The planar embedding of $(C_4 + K_1)^{++-}$ in Figure 6 shows that $(C_4 + K_1)^{++-}$ is planar. Moreover, by Lemma 2.1, $(C_4)^{++-}$ is planar.

The necessity. Let G be a graph of size 4. Then G can be obtained from a graph in Fig. 4 by adding some isolated vertices. By Figure 4, 5, 6, 7 and Lemma 2.1, G^{++-} is nonplanar if $G \notin \{C_4, C_4 + K_1\}$.

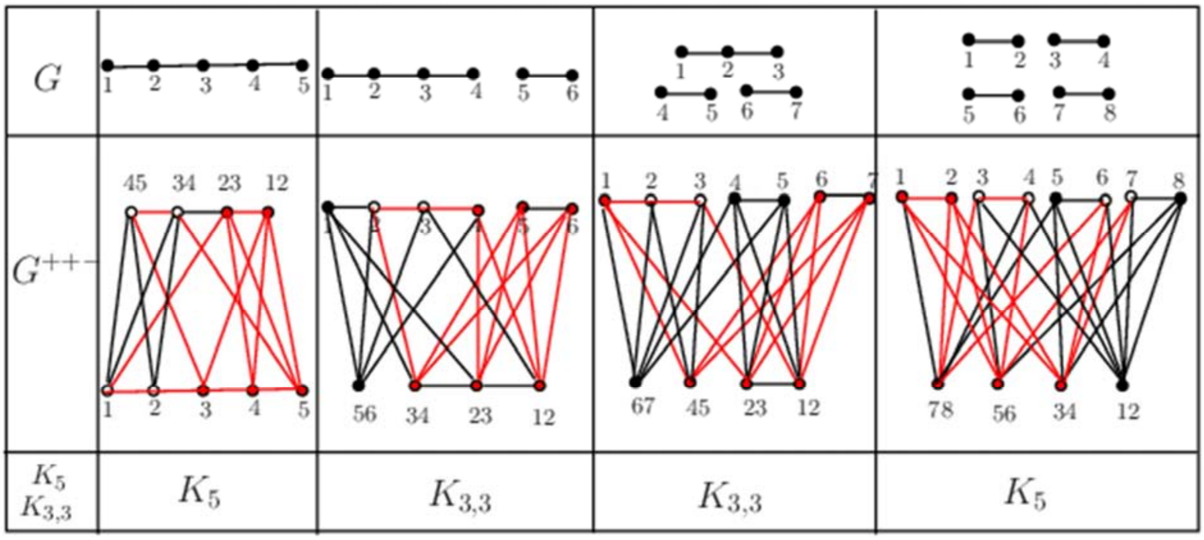

Figure 5. *Transformation graphs G^{++-} of P_5, $P_4 + K_2$, $P_3 + 2K_2$, $4K_2$.*

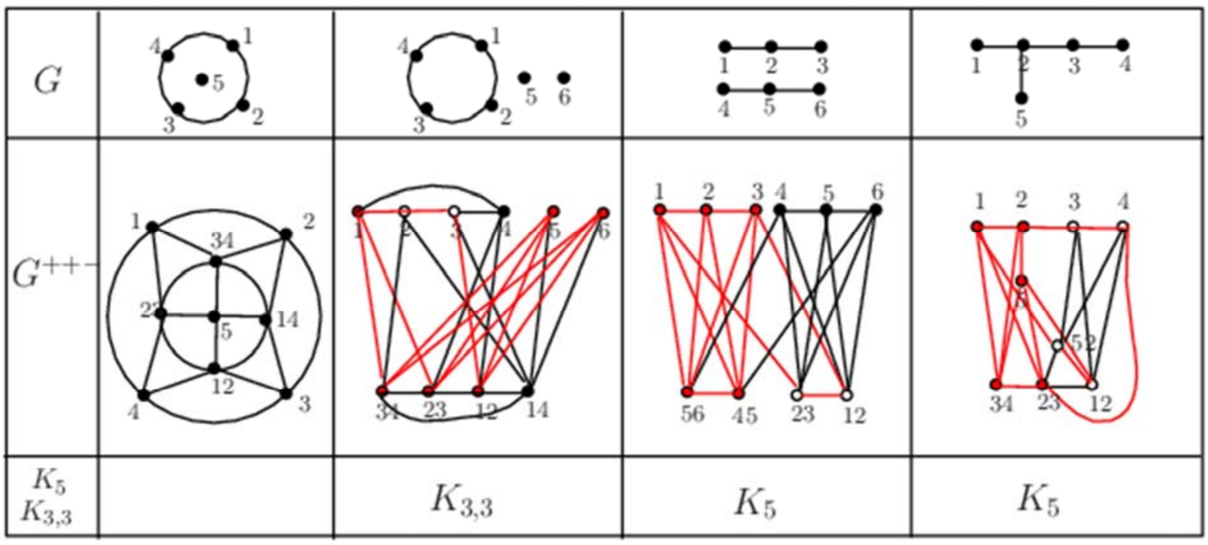

Figure 6. *Transformation graphs G^{++-} of some graphs of size 4.*

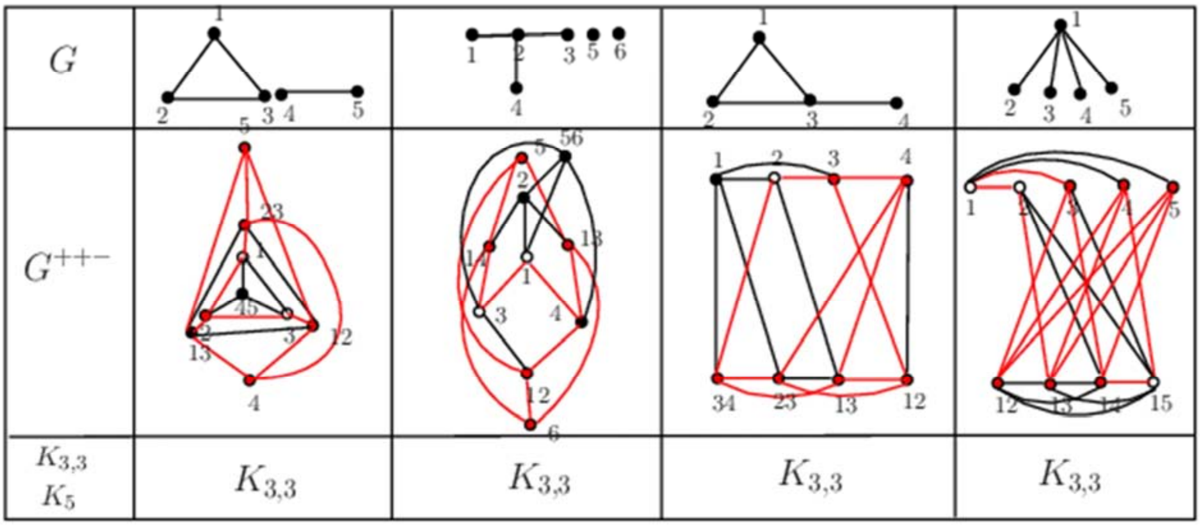

Figure 7. *Transformation graphs G^{++-} of some graphs of size 4.*

Now we consider graphs of size 5. There are precisely twenty six graphs of size 5 without isolated vertices as shown in Figure 8.

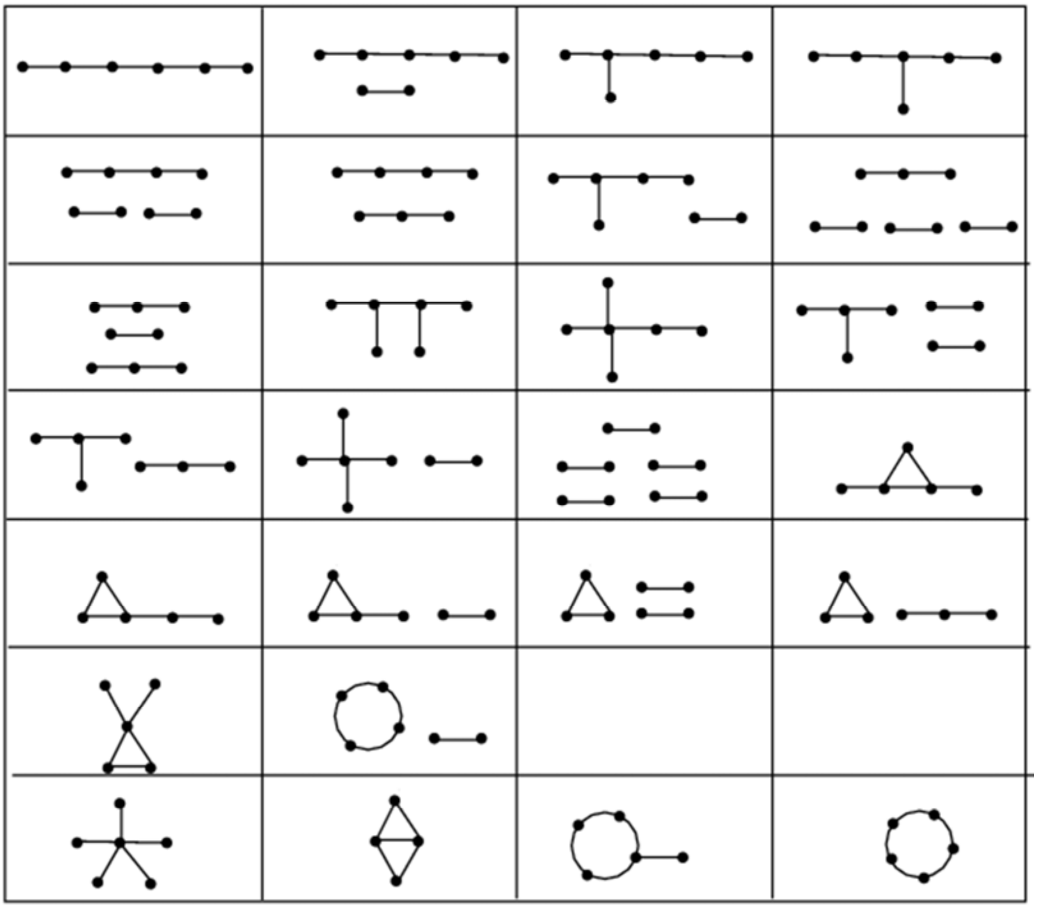

Figure 8. *All graphs of size 5 without isolated vertices.*

Lemma 2.4. For any graph G of size 5, G^{++-} is nonplanar.

Proof. Let G be a graph of size 5, and let H be subgraph of G with size 4 without isoated vertices. By Lemma 2.1, H^{++-} is a subgraph of G^{++-}. By Lemma 2.3, H^{++-} is nonplanar if H is not isomorphic to C_4. Now assume that G contains C_4. Then G is isomorphic to the third graph in Figure 9, and one can see that G^{++-} is nonplanar.

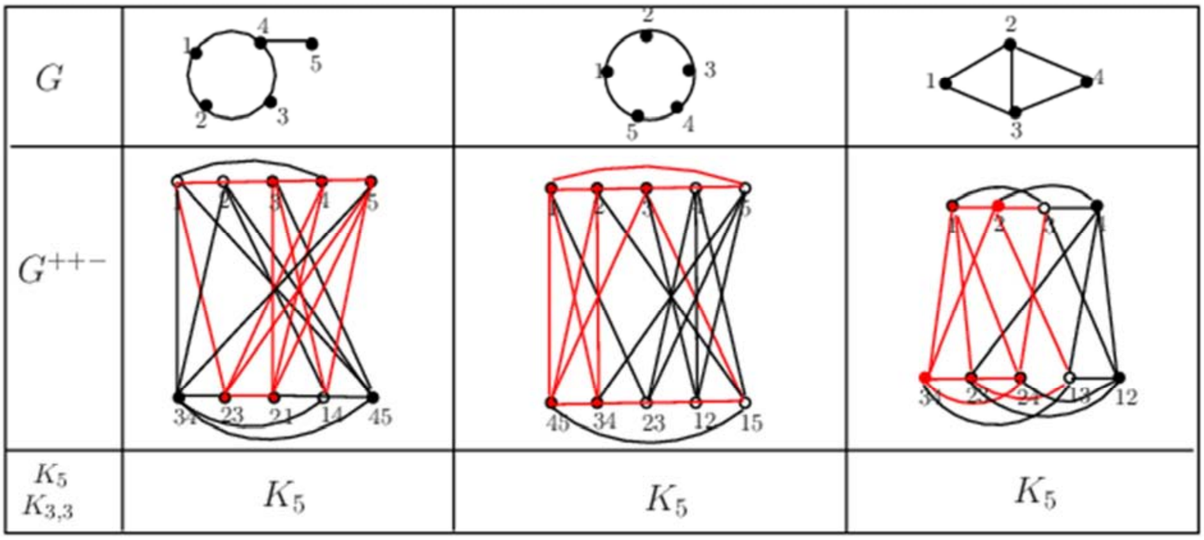

Figure 9. *All graphs of size 5 containing C_4 or C_5, and without induced C_4 and their transformation graphs G^{++-}.*

Lemma 2.5. For a graph G of size m ≥ 6, G^{++-} is nonplanar.

Proof. Trivially, G contains a subgraph H of size 5, and by Lemma 2.1, H^{++-} is a subgraph of G^{++-}. Furthermore, by Lemma 2.4, G^{++-} is nonplanar.

3. Conclusion

In this paper, a necessary and sufficient condition for a graph G such that G^{++-} is planar. It is interesting to investigate some other properties or parameters, such as chromatic number, connectivity, domination number.

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