Application of the Generalized Integral Representation Method (GIRM) to Tidal Wave Propagation

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Abstract: Integral Representation Method (IRM) is one of convenient methods to solve Initial and Boundary Value Problems (IBVP). It can be applied to irregular mesh, and the solution is stable and accurate. IRM is developed to Generalized Integral Representation Method (GIRM) to treat any kinds of problems including nonlinear problems. In GIRM, Generalized Fundamental Solution (GFS) is used instead of Fundamental Solution (FS) in IRM. We can use a variety of GFSs in GIRM. The effects of typical GFSs are investigated. In the present paper, an application of GIRM to tidal wave propagation is discussed, and the time evolution involves the second order time derivatives. An explicit time evolution is used successfully in the present paper.

Keywords: Initial and Boundary Value Problems (IBVP), Generalized Integral Representation Method (GIRM), Generalized Fundamental Solution (GFS), Second Order Time Derivatives in Time Evolution, Tidal Wave Propagation

1. Introduction

Integral Representation Method (IRM) [1-3] is one of convenient methods to solve Initial and Boundary Value Problems (IBVP). It can be applied to irregular mesh, and the solution is stable and accurate. IRM was originally developed to solve linear boundary value problems, and the Fundamental Solution (FS) of the original differential equation plays a major role in IRM. IRM is developed to Generalized Integral Representation Method (GIRM) [4-8] to treat any kinds of problems including nonlinear problems. GIRM uses Generalized Fundamental Solution (GFS). GFS is not required to satisfy the differential equation of the original problem. We can use various GFSs for an individual problem. It is very important to know the characteristics of each GFS.

In the present paper, an application of GIRM to tidal wave propagation is discussed. In this problem, the time evolution involves the second order time derivatives.

2. Initial and Boundary Value Problem (IBVP)

2.1. Shallow Water Wave Problem

In the present paper, a very simple tidal wave problem in horizontal space S with boundary C is discussed:

$$\frac{\partial^2 \eta(\mathbf{x},t)}{\partial t^2} = \nabla \cdot \left(gh(\mathbf{x})\nabla \eta(\mathbf{x},t) \right) \text{ in } S \text{ for } t > 0, \qquad (1)$$

$$\frac{\partial \eta(\mathbf{x},t)}{\partial n} = 0 \text{ on } C \text{ for } t > 0, \qquad (2)$$

$$\eta(\mathbf{x},t) = f(\mathbf{x}), \ \frac{\partial \eta(\mathbf{x},t)}{\partial t} = g(\mathbf{x}) \text{ on } C \text{ at } t = 0, \quad (3a,b)$$

where $\mathbf{x} = (x, y)$ and t are horizontal coordinates and time, respectively. ∇ and $\partial/\partial n$ are operator nabla $(\partial/\partial x, \partial/\partial y)$ and outward normal derivative on C, respectively. $\eta(\mathbf{x}, t)$, $h(\mathbf{x})$ and g are surface elevation, water depth and gravitational acceleration, respectively. $f(\mathbf{x})$ and $g(\mathbf{x})$ are given functions.

2.2. 1-Step Generalized Integral Representation Method (1-Step GIRM)

Let $\tilde{G}(\mathbf{x}, \xi)$ be a Generalized Fundamental Solution (GFS) such as [7,8]

$$\tilde{\eta}(\mathbf{x}, \xi, t) = \tilde{\eta}(\mathbf{x}, \xi) = \frac{1}{2\pi\gamma^2} \exp\left(-\frac{|\mathbf{x} - \xi|^2}{2\gamma^2}\right) \tag{4}$$

From Eq. (1), we have

$$0 = \iint_{S_\mathbf{x}} \left[\frac{\partial^2 \eta(\mathbf{x}, t)}{\partial t^2} - \nabla_\mathbf{x} \cdot (gh(\mathbf{x})\nabla_\mathbf{x}\eta(\mathbf{x}, t)) \right] \tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x}$$

$$= \iint_{S_\mathbf{x}} \frac{\partial^2 \eta(\mathbf{x}, t)}{\partial t^2} \tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x}$$

$$-\iint_{S_\mathbf{x}} \nabla_\mathbf{x} \cdot \left(gh(\mathbf{x})\tilde{G}(\mathbf{x}, \xi)\nabla_\mathbf{x}\eta(\mathbf{x}, t)\right) dS_\mathbf{x}$$

$$+\iint_{S_\mathbf{x}} \left(gh(\mathbf{x})\nabla_\mathbf{x}\eta(\mathbf{x}, t)\right) \cdot \nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x}$$

$$= \iint_{S_\mathbf{x}} \frac{\partial^2 \eta(\mathbf{x}, t)}{\partial t^2} \tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x} \tag{5}$$

$$-\iint_{S_\mathbf{x}} \nabla_\mathbf{x} \cdot \left(gh(\mathbf{x})\tilde{G}(\mathbf{x}, \xi)\nabla_\mathbf{x}\eta(\mathbf{x}, t)\right) dS_\mathbf{x}$$

$$+\iint_{S_\mathbf{x}} \nabla_\mathbf{x} \cdot \left[\eta(\mathbf{x}, t)gh(\mathbf{x})\nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi)\right] dS_\mathbf{x}$$

$$-\iint_{S_\mathbf{x}} \eta(\mathbf{x}, t)\nabla_\mathbf{x} \cdot \left(gh(\mathbf{x})\nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi)\right) dS_\mathbf{x}$$

Rewriting Eq. (5), we obtain

$$\iint_{S_\mathbf{x}} \frac{\partial^2 \eta(\mathbf{x}, t)}{\partial t^2} \tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x}$$

$$= \iint_{S_\mathbf{x}} \eta(\mathbf{x}, t)\nabla_\mathbf{x} \cdot \left(gh(\mathbf{x})\nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi)\right) dS_\mathbf{x}$$

$$+\int_{C_\mathbf{x}} gh(\mathbf{x})\tilde{G}(\mathbf{x}, \xi)\frac{\partial \eta(\mathbf{x}, t)}{\partial n_\mathbf{x}} dC_\mathbf{x} \tag{6}$$

$$-\int_{C_\mathbf{x}} gh(\mathbf{x})\eta(\mathbf{x}, t)\frac{\partial \tilde{G}(\mathbf{x}, \xi)}{\partial n_\mathbf{x}} dC_\mathbf{x}$$

Exchanging $\mathbf{x}$ and ξ, we derive the General Integral Representation (GIR) of Eqs. (1)

$$\iint_{S_\xi} \frac{\partial^2 \eta(\xi, t)}{\partial t^2} \tilde{G}(\xi, \mathbf{x}) dS_\xi$$

$$= \iint_{S_\xi} \left[\eta(\xi, t)\nabla_\xi \cdot \left(gh(\xi)\nabla_\xi\tilde{G}(\xi, \mathbf{x})\right)\right] dS_\xi$$

$$+\int_{C_\xi} gh(\xi)\tilde{G}(\xi, \mathbf{x})\frac{\partial \eta(\xi, t)}{\partial n_\xi} dC_\xi \tag{7}$$

$$-\int_{C_\xi} gh(\xi)\eta(\xi, t)\frac{\partial \tilde{G}(\xi, \mathbf{x})}{\partial n_\xi} dC_\xi$$

Let $\delta(x)$ be Dirac's delta function. If $\nabla_\xi\tilde{G}(\xi, \mathbf{x})$ satisfies

$$\nabla_\xi \cdot \left(gh(\xi)\nabla_\xi\tilde{G}(\xi, \mathbf{x})\right) = \delta(\xi - x)\delta(\eta - y), \tag{8}$$

the first integral on the right-hand side of Eq. (7) becomes $\eta(\mathbf{x}, t)$. Hence, Eq. (7) becomes an integral representation of $\eta(\mathbf{x}, t)$.

If $\eta(x, t)$ at internal point (IP) is known and $\eta(x, t)$ or $\partial\eta(x, t)/\partial n$ at boundary point (BP) is known from the boundary conditions, then the generalized integral representations are integral equations with $\partial^2\eta(x, t)/\partial t^2$ the unknown variables at IP and $\partial\eta(x, t)/\partial n$ or $\eta(x, t)$ the unknown variables at BP. We obtain $\eta(x, t + dt)$ at IP from $\partial\eta(x, t + dt)/\partial t = \partial\eta(x, t)/\partial t + dt\,\partial^2\eta(x, t)/\partial t^2$ and $\eta(x, t + dt) = \eta(x, t) + dt\,\partial\eta(x, t + dt)/\partial t$. If $\eta(x, 0)$ and $\partial\eta(x, 0)/\partial t$ on IP are known from the initial conditions, then we can solve IBVP using the GIR. The GIRs are equivalent to the differential equations.

2.3. 2-Step Generalized Integral Representation Method (2-Step GIRM)

First, we rewrite Eq. (1) as follows.
(a) Non-uniformity equation:

$$\boldsymbol{\theta}(\mathbf{x}, t) = \nabla \eta(\mathbf{x}, t). \tag{9}$$

(b) Constitutive equation:

$$\mathbf{q}(\mathbf{x}, t) = gh(\mathbf{x})\boldsymbol{\theta}(\mathbf{x}, t). \tag{10}$$

(c) Equilibrium equation:

$$\frac{\partial^2 \eta(\mathbf{x}, t)}{\partial t^2} = \nabla \cdot \mathbf{q}(\mathbf{x}, t). \tag{11}$$

From Equations (9), we have

$$0 = \iint_{S_\mathbf{x}} \left[\boldsymbol{\theta}(\mathbf{x}, t) - \nabla_\mathbf{x}\eta(\mathbf{x}, t)\right] \tilde{G}(\mathbf{x}, \xi) dS_\mathbf{x}$$

$$= \iint_{S_\mathbf{x}} \tilde{G}(\mathbf{x}, \xi)\boldsymbol{\theta}(\mathbf{x}, t) dS_\mathbf{x} \tag{12}$$

$$-\iint_{S_\mathbf{x}} \left[\nabla_\mathbf{x}\left(\tilde{G}(\mathbf{x}, \xi)\eta(\mathbf{x}, t)\right) - \eta(\mathbf{x}, t)\nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi)\right] dS_\mathbf{x}$$

Rewriting Eq. (12), we have

$$\iint_{S_\mathbf{x}} \tilde{G}(\mathbf{x}, \xi)\boldsymbol{\theta}(\mathbf{x}, t) dS_\mathbf{x} = -\iint_{S_\mathbf{x}} \left[\eta(\mathbf{x}, t)\nabla_\mathbf{x}\tilde{G}(\mathbf{x}, \xi)\right] dS_\mathbf{x}$$

$$+\int_{C_\mathbf{x}} \tilde{G}(\mathbf{x}, \xi)\eta(\mathbf{x}, t)\mathbf{n}_\mathbf{x}\, dC_\mathbf{x} \tag{13}$$

where $\mathbf{n}$ is the unit outward normal vector of the boundary C. Exchanging $\mathbf{x}$ and ξ, we obtain

$$\iint_{S_\xi} \tilde{G}(\xi, \mathbf{x})\boldsymbol{\theta}(\xi, t) dS_\xi = -\iint_{S_\xi} \left[\eta(\xi, t)\nabla_\xi\tilde{G}(\xi, \mathbf{x})\right] dS_\xi$$

$$+\int_{C_\xi} \tilde{G}(\xi, \mathbf{x})\eta(\xi, t)\mathbf{n}_\xi\, dC_\xi \tag{14}$$

From Eq. (11), we have

$$0 = \iint_{S_x}\left[\frac{\partial^2\eta(\mathbf{x},t)}{\partial t^2} - \nabla_\mathbf{x}\cdot\mathbf{q}(\mathbf{x},t)\right]\tilde{G}(\mathbf{x},\xi)\,dS_\mathbf{x}$$

$$= \iint_{S_x}\tilde{G}(\mathbf{x},\xi)\frac{\partial^2\eta(\mathbf{x},t)}{\partial t^2}\,dS_\mathbf{x} \qquad (15)$$

$$-\iint_{S_x}\nabla_\mathbf{x}\cdot\left(\tilde{G}(\mathbf{x},\xi)\mathbf{q}(\mathbf{x},t)\right)dS_\mathbf{x} + \iint_{S_x}\nabla_\mathbf{x}\tilde{G}(\mathbf{x},\xi)\cdot\mathbf{q}(\mathbf{x},t)\,dS_\mathbf{x}$$

Rewriting Eq. (15), we obtain

$$\iint_{S_x}\tilde{G}(\mathbf{x},\xi)\frac{\partial^2\eta(\mathbf{x},t)}{\partial t^2}\,dS_\mathbf{x}$$

$$= -\iint_{S_x}\nabla_\mathbf{x}\tilde{G}(\mathbf{x},\xi)\cdot\mathbf{q}(\mathbf{x},t)\,dS_\mathbf{x} + \int_{C_x}\tilde{G}(\mathbf{x},\xi)\mathbf{q}(\mathbf{x},t)\cdot\mathbf{n}_\mathbf{x}\,dC_\mathbf{x} \qquad (16)$$

Exchanging $\mathbf{x}$ and ξ in Eq. (16), we derive

$$\iint_{S_\xi}\tilde{G}(\xi,\mathbf{x})\frac{\partial^2\eta(\xi,t)}{\partial t^2}\,dS_\xi$$

$$= -\iint_{S_\xi}\nabla_\xi\tilde{G}(\xi,\mathbf{x})\cdot\mathbf{q}(\xi,t)\,dS_\xi + \int_{C_\xi}\tilde{G}(\xi,\mathbf{x})\mathbf{q}(\xi,t)\cdot\mathbf{n}_\xi\,dC_\xi \qquad (17)$$

Equations (14) and (17) are GIRs of Eqs. (9) and (11), respectively.

If $\eta(\mathbf{x},t)$ at IP is known, then GIR given by Eq. (14) is an integral equation with $\theta(\mathbf{x},t)$ the unknown variable at IP and $\eta(\mathbf{x},t)$ the unknown variables at BP. These unknown variables are obtained solving the integral equation. $\mathbf{q}(\mathbf{x},t)$ at IP is obtained from Equation (10). Then, GIR given by Eq. (17) is an integral equation with $\partial^2\eta(\mathbf{x},t)/\partial t^2$ the unknown variable at IP and $\mathbf{q}(\mathbf{x},t)$ the unknown variables at BP. These unknown variables are obtained solving the integral equation. We obtain $\partial\eta(\mathbf{x},t+dt)/\partial t$ and $\eta(\mathbf{x},t+dt)$ at IP from $\partial\eta(\mathbf{x},t+dt)/\partial t = \partial\eta(\mathbf{x},t)/\partial t + dt\,\partial^2\eta(\mathbf{x},t)/\partial t^2$ and $\eta(\mathbf{x},t+dt) = \eta(\mathbf{x},t) + dt\,\partial\eta(\mathbf{x},t+dt)/\partial t$, respectively. We repeat the procedure mentioned above at every time step. If $\eta(x,0)$ and $\partial\eta(x,0)/\partial t$ on IP are known from the initial conditions, then we can solve IBVP using GIRs. GIRs are equivalent to the differential equations.

3. Numerical Results

For simplicity, we focus on one-dimensional problem in region $-L < x < +L$ given by

$$\frac{\partial^2\eta(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(gh(x)\frac{\partial\eta(x,t)}{\partial x}\right),$$
$$in -L < x < +L \ for \ t > 0 \qquad (18)$$

$$\frac{\partial\eta(+L,t)}{\partial x} = 0, \quad \frac{\partial\eta(-L,t)}{\partial x} = 0 \ \text{ for } t > 0, \qquad (19a,b)$$

$$\eta(x,0) = f(x), \frac{\partial\eta(x,0)}{\partial t} = g(x)$$
$$in -L < x < +L \qquad (20a,b)$$

We use four kinds of GFSs [8]:
(1) C^0 Continuous GFS
Harmonic GFS

$$\tilde{G}(x,\xi) = \frac{1}{2}|x-\xi|, \qquad (21a)$$

$$\frac{\partial\tilde{G}(x,\xi)}{\partial x} = \frac{1}{2}\text{sgn}(x-\xi), \qquad (21b)$$

$$\frac{\partial^2\tilde{G}(x,\xi)}{\partial x^2} = \delta(x-\xi), \qquad (21c)$$

Exponential GFS

$$\tilde{G}(x,\xi) = -\frac{\gamma}{2}\exp\left(-\frac{|x-\xi|}{\gamma}\right), \qquad (22a)$$

$$\frac{\partial\tilde{G}(x,\xi)}{\partial x} = \frac{1}{2}\exp\left(-\frac{|x-\xi|}{\gamma}\right)\text{sgn}(x-\xi), \qquad (22b)$$

$$\frac{\partial^2\tilde{G}(x,\xi)}{\partial x^2} = -\frac{1}{2\gamma}\exp\left(-\frac{|x-\xi|}{\gamma}\right) + \delta(x-\xi). \qquad (22c)$$

(2) C^2 Continuous GFS [8-10]
Lucy GFS

$$\tilde{G}(x,\xi) = \begin{cases}\dfrac{5}{4}\dfrac{1}{\gamma}\left(1+3\dfrac{|x-\xi|}{\gamma}\right)\left(1-\dfrac{|x-\xi|}{\gamma}\right)^3, & |x-\xi|\le\gamma\\[2mm] 0, & |x-\xi|>\gamma\end{cases} \qquad (23a)$$

$$\frac{\partial\tilde{G}(x,\xi)}{\partial x} = \begin{cases}\left[\dfrac{5}{4}\dfrac{3}{\gamma^2}\left(1-\dfrac{|x-\xi|}{\gamma}\right)^3 - \dfrac{5}{4}\dfrac{3}{\gamma^2}\left(1+3\dfrac{|x-\xi|}{\gamma}\right)\left(1-\dfrac{|x-\xi|}{\gamma}\right)^2\right]\cdot\\[2mm] \quad\quad\cdot\text{sgn}(x-\xi),\ |x-\xi|\le\gamma\\[2mm] 0, \quad\quad |x-\xi|>\gamma\end{cases} \qquad (23b)$$

$$\frac{\partial^2\tilde{G}(x,\xi)}{\partial x^2} = \begin{cases}\left[-\dfrac{5}{4}\dfrac{9}{\gamma^3}\left(1-\dfrac{|x-\xi|}{\gamma}\right)^2 + \dfrac{5}{4}\dfrac{6}{\gamma^3}\left(1+3\dfrac{|x-\xi|}{\gamma}\right)\left(1-\dfrac{|x-\xi|}{\gamma}\right)\right],\\[2mm] \quad\quad |x-\xi|\le\gamma\\[2mm] 0, \quad\quad |x-\xi|>\gamma\end{cases} \qquad (23c)$$

(3) C^∞ Continuous GFS
Gaussian GFS

$$\tilde{G}(x,\xi) = \frac{1}{\sqrt{2\pi}\,\gamma}\exp\left(-\frac{(x-\xi)^2}{2\gamma^2}\right), \qquad (24a)$$

$$\frac{\partial\tilde{G}(x,\xi)}{\partial x} = -\frac{1}{\sqrt{2\pi}\,\gamma^3}(x-\xi)\exp\left(-\frac{(x-\xi)^2}{2\gamma^2}\right), \qquad (24b)$$

$$\frac{\partial^2\tilde{G}(x,\xi)}{\partial x^2} = -\frac{1}{\sqrt{2\pi}\,\gamma^3}\exp\left(-\frac{(x-\xi)^2}{2\gamma^2}\right)$$
$$+ \frac{1}{\sqrt{2\pi}\,\gamma^5}(x-\xi)^2\exp\left(-\frac{(x-\xi)^2}{2\gamma^2}\right) \qquad (24c)$$

3.1. Travelling of a Wave in Infinite One-Dimensional Space

For this problem, we use One-Step GIRM. We rewrite Eq. (7) for one-dimensional case:

$$\int_{-L}^{+L} \frac{\partial^2 \eta(\xi,t)}{\partial t^2} \tilde{G}(\xi,x)\,d\xi = \int_{-L}^{+L} \eta(\xi,t) \frac{\partial}{\partial \xi}\left(gh(\xi) \frac{\partial \tilde{G}(\xi,x)}{\partial \xi} \right)d\xi$$

$$+ \left[gh(+L)\tilde{G}(+L,x)\frac{\partial \eta(+L,t)}{\partial \xi} - gh(-L)\tilde{G}(-L,x)\frac{\partial \eta(-L,t)}{\partial \xi} \right]. \quad (25)$$

$$- \left[gh(+L)\eta(+L,t)\frac{\partial \tilde{G}(+L,x)}{\partial \xi} - gh(-L)\eta(-L,t)\frac{\partial \tilde{G}(-L,x)}{\partial \xi} \right]$$

If we consider infinite one-dimensional space, L is taken large enough, and we assume

$$\frac{\partial C(\pm L,t)}{\partial x}=0 \quad \text{and} \quad C(\pm L,t)=0. \quad (26)$$

Equation (25) is now approximated by

$$\int_{-L}^{+L} \frac{\partial^2 \eta(\xi,t)}{\partial t^2} \tilde{G}(\xi,x)\,d\xi = \int_{-L}^{+L} \eta(\xi,t) \frac{\partial}{\partial \xi}\left(gh(\xi) \frac{\partial \tilde{G}(\xi,x)}{\partial \xi} \right)d\xi. \quad (27)$$

The time evolution with time step dt is given by

$$\frac{\partial \eta(x,t+dt)}{\partial t} \approx \frac{\partial \eta(x,t)}{\partial t} + \frac{\partial^2 \eta(x,t)}{\partial t^2} dt \quad (28a)$$

$$\eta(x,t+dt) \approx \eta(x,t) + \frac{\partial \eta(x,t+dt)}{\partial t} dt \quad (28b)$$

The initial value is specified as

$$\eta(x,0) = \exp\left(-\frac{x^2}{2(L/8)^2} \right), \quad -L < x < L. \quad (29)$$

The discretization of Eq. (27) is briefly discussed in Appendix A [7,8]. The solution of the Initial Boundary Value Problem (IBVP) is summarized as

From Eq. (27), $\eta(\mathbf{x},t) \rightarrow \partial^2 \eta(\mathbf{x},t)/\partial t^2$;

from Eq. (28a), $\partial^2 \eta(\mathbf{x},t)/\partial t^2 \rightarrow \partial \eta(\mathbf{x},t+dt)/\partial t$;

from Eq. (28b), $\partial \eta(\mathbf{x},t+dt)/\partial t \rightarrow \eta(\mathbf{x},t+dt)$;

repeat. (30)

Parameters for numerical calculations are given below:

$$c = \sqrt{gh} = 2; \quad L = 4; \quad N = 80;$$

$$dx = 2L/N = 0.1; \quad dt = 0.001, \quad (31)$$

where N is the number of elements in the region $-L < x < L$.

The exact solution is given in Fig. 1, and the numerical results are given in Fig. 2. The accurate results are obtained among numerical results using Harmonic, Exponential and Gaussian GFSs. The numerical result using Lucy GFS is not accurate enough.
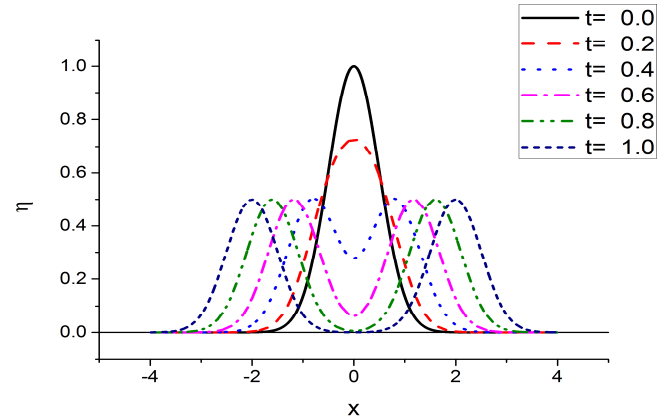

Figure 1. Exact solution of $\eta(x,t)$

3.2. Reflection of a Wave by Walls at Boundaries

For this problem, we use Two-Step GIRM. We rewrite Eqs. (14) and (17) for one-dimensional case:

$$\int_{-L}^{+L} \tilde{G}(\xi,x)\theta(\xi,t)\,d\xi$$
$$= -\int_{-L}^{+L} \eta(\xi,t) \frac{\partial \tilde{G}(\xi,x)}{\partial \xi}\,d\xi \quad (32)$$
$$+ \left[\tilde{G}(+L,x)\eta(+L,t) - \tilde{G}(-L,x)\eta(-L,t) \right]$$

$$\int_{-L}^{+L} \tilde{G}(\xi,x) \frac{\partial^2 \eta(\xi,t)}{\partial t^2}\,d\xi$$
$$= -\int_{-L}^{+L} \frac{\partial \tilde{G}(\xi,x)}{\partial \xi} q(\xi,t)\,d\xi \quad (33)$$
$$+ \left[\tilde{G}(+L,x)q(+L,t) - \tilde{G}(-L,x)q(-L,t) \right]$$

The boundary values $\eta(+L,t)$ and $\eta(-L,t)$ are approximated, for example, by

$$\eta(+L,t) \approx \eta(x_{N-1},t), \quad \eta(-L,t) \approx \eta(x_0,t), \quad (34)$$

where x_i is given by $x_i = -L + (i+0.5)L/N$, $i = 0, 1, \cdots, N-1$. Then, Eq. (32) rewritten as

$$\int_{-L}^{+L} \tilde{G}(\xi,x)\theta(\xi,t)\,d\xi$$
$$= -\int_{-L}^{+L} \eta(\xi,t) \frac{\partial \tilde{G}(\xi,x)}{\partial \xi}\,d\xi \quad (35)$$
$$+ \left[\tilde{G}(+L,x)\eta(x_{N-1},t) - \tilde{G}(-L,x)\eta(x_0,t) \right]$$

And substituting the boundary condition (19), Eq. (33) becomes

$$\int_{-L}^{+L} \tilde{G}(\xi,x) \frac{\partial^2 \eta(\xi,t)}{\partial t^2}\,d\xi = -\int_{-L}^{+L} \frac{\partial \tilde{G}(\xi,x)}{\partial \xi} q(\xi,t)\,d\xi. \quad (36)$$

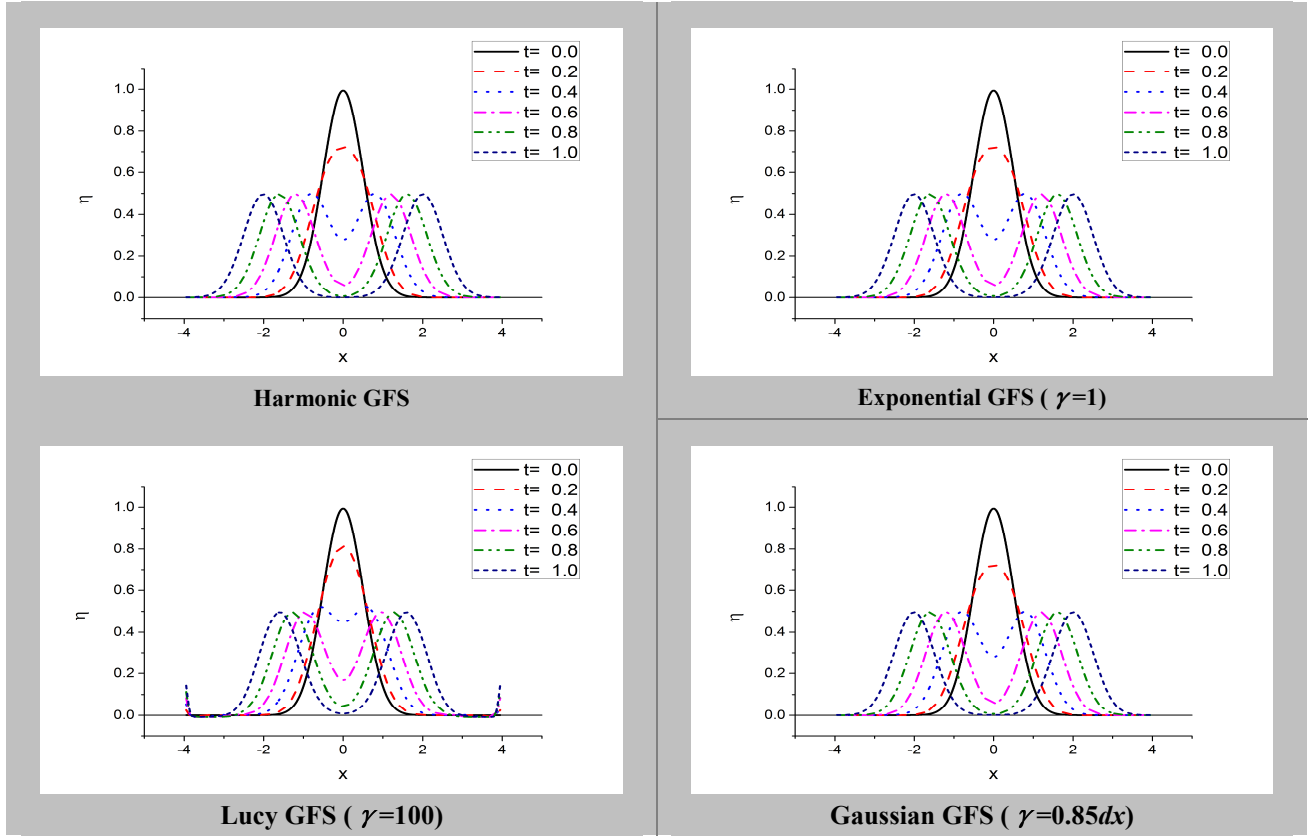The same time evolution (28) and initial condition (29) are   used.

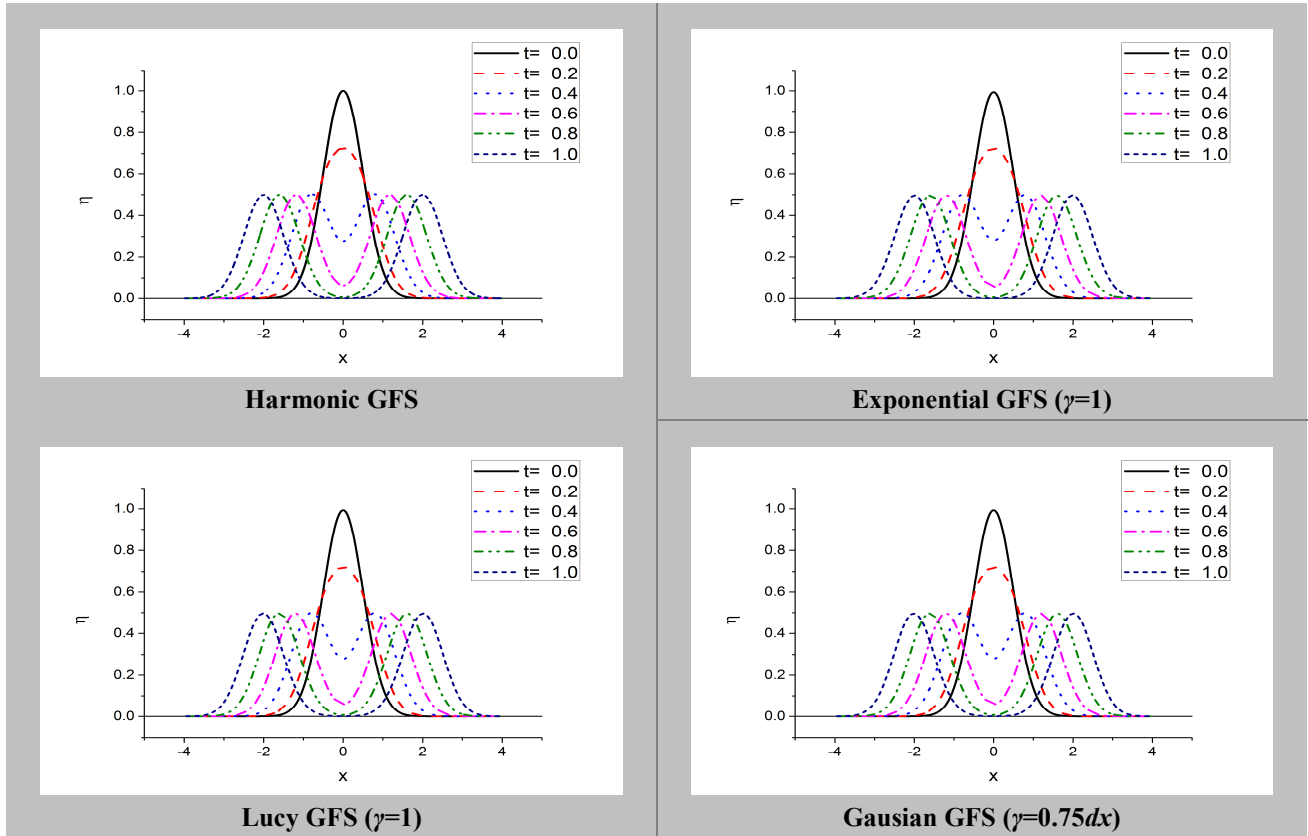**Figure 2.** *Numerical results using various GFSs*

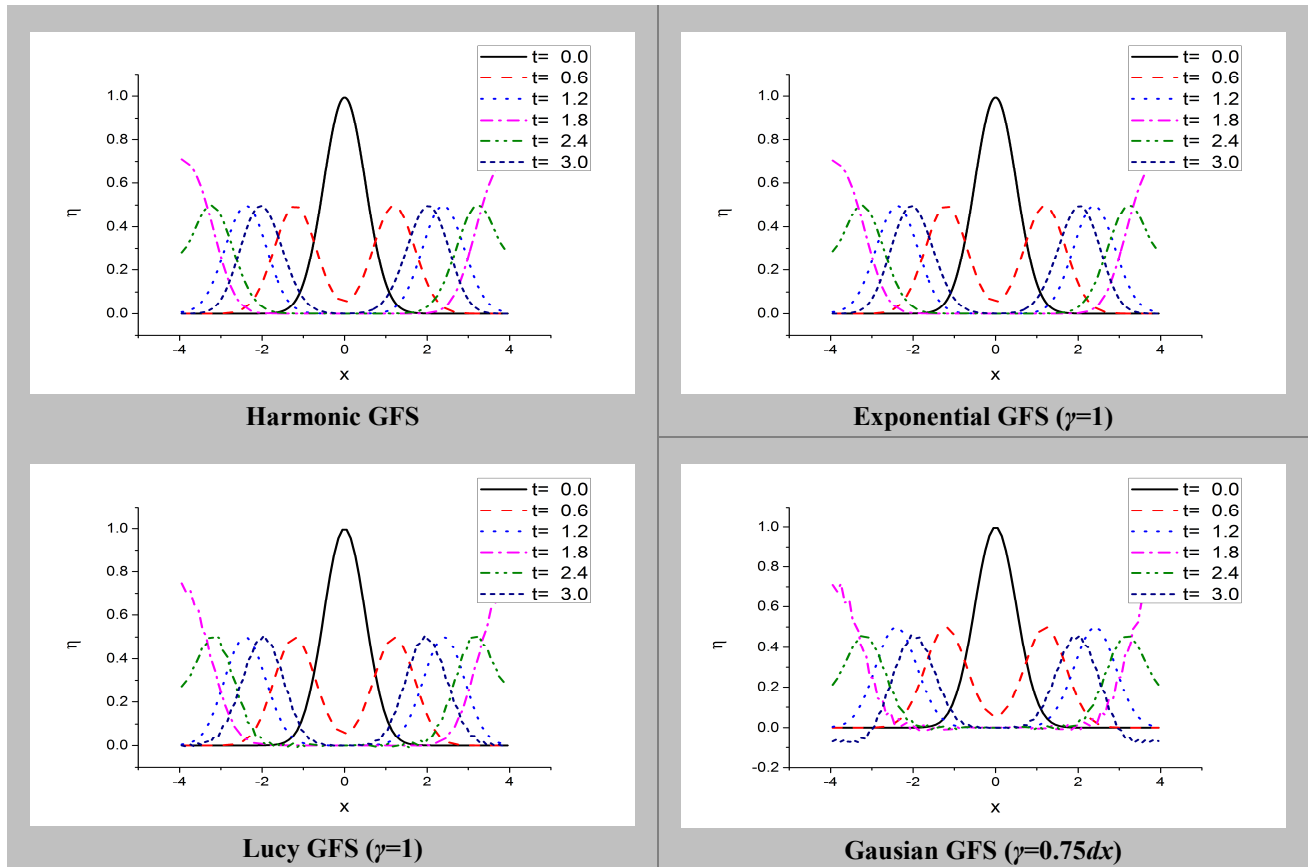**Figure 3.** *Without reflection at boundaries*

Figure 4. *With reflection at boundaries*

The discretization of Eqs. (35) and (36) is discussed in Ref. [7,8]. The solution of the Initial Boundary Value Problem (IBVP) is summarized as

From Eq. (35), $\eta(\mathbf{x},t) \to \boldsymbol{\theta}(\mathbf{x},t)$;

from (10), $\boldsymbol{\theta}(\mathbf{x},t) \to \mathbf{q}(\mathbf{x},t)$;

from (36), $\mathbf{q}(\mathbf{x},t) \to \partial^2 \eta(\mathbf{x},t)/\partial t^2$;

from Eq. (28a), $\partial^2 \eta(\mathbf{x},t)/\partial t^2 \to \partial \eta(\mathbf{x},t+dt)/\partial t$;

from Eq. (28b), $\partial \eta(\mathbf{x},t+dt)/\partial t \to \eta(\mathbf{x},t+dt)$;

repeat.   (37)

For numerical calculations, same parameters given by Eq. (31) are used. The numerical results in Fig. 3 do not include reflection by walls, and the accurate results are obtained for all cases using Harmonic, exponential, Lucy and Gaussian GFSs. However, for cases involving reflection at walls as shown in Fig. 4, accurate numerical results are obtained only when Harmonic and Exponential GFSs are used. Small spurious oscillations are involved in numerical results using Lucy GFS. The numerical result using Gaussian GFS contains errors after reflection. This may correspond to our experience with respect to non-zero boundary values discussed in Ref. [8].

4. Conclusions

A set of integral representations is obtained using a fundamental solution of a differential-type boundary value problem [1-3]. If the boundary conditions are substituted into the set of the integral representations, a set of integral equations is obtained. The unknown variables of the boundary value problem can be determined by solving the set of integral equations derived from the set of integral representations.

Furthermore, we proposed a Generalized Integral Representation Method (GIRM) [4-8]. In GIRM, Generalized Fundamental Solution (GFS) is used. In the present paper, GIRM was applied to propagation of tidal wave. Unlike Integral Representation Method (IRM), we can use a variety of GFSs in GIRM. Four kinds of GFSs, namely Harmonic, Exponential, Lucy and Gaussian GFSs, were used. When the reflection of waves did not take place, all GFSs gave accurate numerical results. However, Lucy and Gaussian GFSs did not give reflected waves correctly enough. This may correspond to our experience with respect to non-zero boundary values discussed in Ref. [8].

In case of wave propagation discussed in the present paper, the second order time derivatives appear in the time evolution. An explicit time evolution is used successfully in the present paper.

Appendix A. Discretization of Eq. (27)

The region $-L < x < +L$ is divided into N intervals:

$$dx = 2L/N \, , \tag{A1}$$

$$x_i = \xi_i = -L + (i+0.5)dx \, , \quad i = 0, 1, \cdots, N-1 \, , \tag{A2}$$

Equation (27) is discretized as follows:

$$\sum_{j=0}^{N} \left[\frac{\partial^2 \eta(\xi,t)}{\partial t^2} \right]_{\xi=\xi_j} \int_{x_j+dx/2}^{x_j+dx/2} \tilde{G}(\xi, x_i) \, d\xi$$
$$= \sum_{j=0}^{N} \eta(\xi_j, t) \int_{x_j-dx/2}^{x_j+dx/2} \frac{\partial}{\partial \xi} \left(gh(\xi) \frac{\partial \tilde{G}(\xi, x_i)}{\partial \xi} \right) d\xi \tag{A3}$$

The integrals in Eq. (A2) can be evaluated numerically.

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