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Inequalities for the Mixed Radial Blaschke-Minkowski Homomorphisms and the Applications

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Abstract: The notion of intersection body is introduced by Lutwak in 1988, it is one of important research contents and led to the studies of Busemann-Petty problem in the Brunn-Minkowski theory. Based on the properties of the intersection bodies, Schuster introduced the notion of radial Blaschke-Minkowski homomorphisms and proved a lot of related inequalities. In this paper, by applying the dual mixed volume theory and analytic inequalities, we first give a lower bound of the dual quermassintegrals for the mixed radial Blaschke-Minkowski homomorphisms. As its an application, we get a reverse form of the well-known Busemann intersection inequality. Further, a Brunn-Minkowski type inequality of the L_p radial Minkowski sum for the dual quermassintegrals of mixed radial Blaschke-Minkowski homomorphisms is established, and then the intersection body version of this Brunn-Minkowski type inequality is yielded. From this, we not only extend Schuster's related result but also obtain the Brunn-Minkowski type inequalities of L_p harmonic radial sum and L_p radial Blaschke sum, respectively.

Keywords: Dual Quermassintegral, Intersection Body, Radial Blaschke-Minkowski Homomorphism, Busemann Intersection Inequality, L_p Radial Minkowski Sum

1. Introduction

The setting for this paper is Euclidean n-space $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of body K. For the centered unit n-ball B, write $V(B)=\omega_n$.

If K is a compact star-shaped (about the origin) in $\mathbb{E}^n$, its radial function, $\rho_K=\rho(K,\cdot)$, is defined by [1]

$$\rho(K,u)=\max\{\lambda\geq 0:\lambda u\in K\}$$

for all $u\in S^{n-1}$. If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of all star bodies (about the origin) in $\mathbb{R}^n$.

The notion of intersection body was explicitly defined by Lutwak [2]. For $K\in\mathcal{S}_o^n$ and $n\geq 2$, the intersection body, IK, of K as a star body whose radial function in direction $u\in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by the hyperplane orthogonal to u, i.e.,

$$\rho(IK,u)=v(K\cap u^{\perp}),$$

where v is the $n-1$-dimensional volume and $u^{\perp}$ is the $n-1$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $u\in S^{n-1}$.

Intersection body is one of the central notions and led to the studies of Busemann-Petty problem in the Brunn-Minkowski theory [1, 3, 4, 2, 5, 6, 7, 8, 9]. Based on the properties of intersection bodies, Schuster [10] introduced the radial Blaschke-Minkowski homomorphisms as follows:

Definition 1.A A map $\Psi:\mathcal{S}_o^n\to\mathcal{S}_o^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(1) Ψ is continuous.

(2) ΨK is radial Blaschke-Minkowski additive. i.e. $\Psi(K\widehat{+}L)=\Psi K\widetilde{+}\Psi L$ for all $K,L\in\mathcal{S}_o^n$. Here $K\widehat{+}L$ dentes the radial Blaschke sum of K and L, and $\Psi K\widetilde{+}\Psi L$ denotes the radial Minkowski sum of ΨK and ΨL.

(3) Ψ intertwines rotations, i.e. $\Psi(\vartheta K)=\vartheta\Psi K$ for all $K\in\mathcal{S}_o^n$ and all $\vartheta\in SO(n)$.

According to above definition, Schuster [10] proved the following important result:

Theorem 1.A There is a continuous operator

$$\Psi : \underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n-1} \to \mathcal{S}_o^n$$

symmetric in its arguments such that, for $L_1, \cdots, L_m \in \mathcal{S}_o^n$ and $\lambda_1, \cdots, \lambda_m \geq 0$,

$$\Psi(\lambda_1 L_1 \widetilde{+} \cdots \widetilde{+} \lambda_m L_m) = \widetilde{\sum_{i_1, \cdots, i_{n-1}}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(L_{i_1}, \cdots, L_{i_{n-1}}),$$

where the sum is with respect to radial Minkowski addition.

Theorem 1.A generalizes the notion of radial Blaschke Minkowski homomorphism. We call

$$\Psi : \underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n-1} \to \mathcal{S}_o^n$$

the mixed radial Blaschke-Minkowski homomorphisms denoted by $\Psi(K_1, K_2, \cdots, K_{n-1})$ for $K_1, K_2, \cdots, K_{n-1} \in \mathcal{S}_o^n$.

For $K, L \in \mathcal{S}_o^n$, let $\Psi_i(K, L) = \Psi(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i})$ $(i = 0, 1, \cdots, n-1)$ denote the mixed radial Blaschke-Minkowski homomorphisms of K and L. If $L = B$, we write $\Psi_i K = \Psi_i(K, B)$ and call $\Psi_i K$ the mixed radial Blaschke-Minkowski homomorphisms of K. Clearly, $\Psi_0 K = \Psi K$.

Whereafter, Schuster [11] considered the Busemann-Petty problem for the radial Blaschke-Minkowski homomorphisms. In 2011, Wang, Liu and He [12] extended Schuster's radial Blaschke-Minkowski homomorphisms to L_p analogies. In recent years, a lot of important conclusions for the radial Blaschke-Minkowski homomorphisms and their L_p analogies were obtained, see e.g. [13, 14, 15, 16, 17, 12, 18, 19, 20, 21, 22, 23, 24, 25].

One purpose of this paper is to establish a lower bound for the dual quermassintegrals of the mixed radial Blaschke-Minkowski homomorphisms:

Theorem 1.1 If $K \in \mathcal{S}_o^n$, real $i < n-1$ and $j = 0, 1, \cdots, n-1$, then

$$\widetilde{W}_i(\Psi_j K) \geq \frac{r_\Psi^{n-i}}{\omega_n^{n-i-1}} \widetilde{W}_{j+1}(K)^{n-i},$$

with equality if and only if $\Psi_j K$ is a ball. Here r_Ψ denotes the radius of ΨB, $\widetilde{W}_i(M)$ denotes the dual quermassintegrals of $M \in \mathcal{S}_o^n$.

Let $i = j = 0$ in Theorem 1.1, and notice that $\Psi_0 K = \Psi K$ and $\widetilde{W}_0(M) = V(M)$, we obtain that

Corollary 1.1 If $K \in \mathcal{S}_o^n$, then

$$V(\Psi K) \geq \frac{r_\Psi^n}{\omega_n^{n-1}} \widetilde{W}_1(K)^n,$$

with equality if and only if ΨK is a ball. Here r_Ψ denotes the radius of ΨB.

Since the intersection operator I is an example of a radial Blaschke-Minkowski homomorphism, thus, together with the radius of IB is ω_{n-1}, Corollary 1.1 provides the following a new inequality for the volume of intersection body.

Corollary 1.2 If $K \in \mathcal{S}_o^n$, then

$$V(IK) \geq \frac{\omega_{n-1}^n}{\omega_n^{n-1}} \widetilde{W}_1(K)^n, \tag{1}$$

with equality if and only if IK is a ball.

Remark 1.1 The fundamental volume inequality for intersection bodies is the well-known Busemann intersection inequality [26, 1]: *If K is a convex body (i.e. compact, convex subsets with non-empty interiors) containing the origin in its interiors in $\mathbb{R}^n$, then*

$$V(IK) \leq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} V(K)^{n-1}, \tag{2}$$

with equality if and only if K is a centered ellipsoid. Compare to inequality (1) and inequality (2), inequality (1) may be regards as a reverse form of Busemann intersection inequality.

In [10], Schuster proved a lot of inequalities for mixed radial Blaschke-Minkowski homomorphisms, especially, the following Brunn-Minkowski type inequality of radial sum of star bodies for mixed radial Blaschke-Minkowski homomorphisms was given.

Theorem 1.B If $K, L \in \mathcal{S}_o^n$, $i = 0, 1, \cdots, n-2$ and $j = 0, 1, \cdots, n-2$, then

$$\widetilde{W}_i(\Psi_j(K \widetilde{+} L))^{\frac{1}{(n-i)(n-j-1)}} \leq \widetilde{W}_i(\Psi_j K)^{\frac{1}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \tag{3}$$

with equality if and only if K and L are dilates. Here, $K \widetilde{+} L$ denotes the radial sum of K and L.

Our next result generalizes the above Brunn-Minkowski type inequality to L_p radial Minkowski sum of star bodies.

Theorem 1.2 Let $K, L \in \mathcal{S}_o^n$, real $p \neq 0$, $i = 0, 1, \cdots, n-2$ and j is an integer. If $p < 0$ and $0 \leq j < n-1$, then

$$\widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))^{\frac{p}{(n-i)(n-j-1)}} \geq \widetilde{W}_i(\Psi_j K)^{\frac{p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{p}{(n-i)(n-j-1)}}; \tag{4}$$

if $0<p<n-1$ and $0\le j<n-p-1$, then

$$\widetilde{W}_i(\Psi_j(K\widetilde{+}_pL))^{\frac{p}{(n-i)(n-j-1)}}\le\widetilde{W}_i(\Psi_jK)^{\frac{p}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_jL)^{\frac{p}{(n-i)(n-j-1)}}. \tag{5}$$

In each inequality, equality holds if and only if K and L are dilates. Here, $K\widetilde{+}_pL$ denotes the L_p-radial sum of K and L.

Remark 1.2 If $p=1$ in Theorem 1.2, then inequality (4) just is inequality (3). Let $i=j=0$ in Theorem 1.2, and notice intersection operator I is a special case of radial Blaschke-Minkowski homomorphisms, we have

Corollary 1.3 Let $K,L\in\mathcal{S}_o^n$, real $p\neq 0$. If $p<0$, then

$$V(I(K\widetilde{+}_pL))^{\frac{p}{n(n-1)}}\ge V(IK)^{\frac{p}{n(n-1)}}+V(IL)^{\frac{p}{n(n-1)}};$$

if $0<p<n-1$, then

$$V(I(K\widetilde{+}_pL))^{\frac{p}{n(n-1)}}\le V(IK)^{\frac{p}{n(n-1)}}+V(IL)^{\frac{p}{n(n-1)}}.$$

In each inequality, equality holds if and only if K and L are dilates.

Remark 1.3 Let $p=-q$ ($q\ge 1$) or $p=n-q$ ($1<q<n$) in Theorem 1.2, respectively, then inequality (4) orderly gives the Brunn-Minkowski type inequalities for the L_q harmonic radial sum and the L_q radial Blaschke sum as follows:

Corollary 1.4 If $K,L\in\mathcal{S}_o^n$, real $q\ge 1$, $i=0,1,\cdots,n-2$ and $j=0,1,\cdots,n-2$, then

$$\widetilde{W}_i(\Psi_j(K\widetilde{+}_{-q}L))^{-\frac{q}{(n-i)(n-j-1)}}\ge\widetilde{W}_i(\Psi_jK)^{-\frac{q}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_jL)^{-\frac{q}{(n-i)(n-j-1)}},$$

with inequality if and only if K and L are dilates. Here, $K\widetilde{+}_{-q}L$ denotes the L_q harmonic radial sum of K and L.

Corollary 1.5 If $K,L\in\mathcal{S}_o^n$, real $1<q<n$, $i=0,1,\cdots,n-2$ and integer j satisfies $0\le j<q-1$, then

$$\widetilde{W}_i(\Psi_j(K\widehat{+}_qL))^{\frac{n-q}{(n-i)(n-j-1)}}\le\widetilde{W}_i(\Psi_jK)^{\frac{n-q}{(n-i)(n-j-1)}}+\widetilde{W}_i(\Psi_jL)^{\frac{n-q}{(n-i)(n-j-1)}},$$

with inequality if and only if K and L are dilates. Here, $K\widehat{+}_qL$ denotes the L_q-radial Blaschke sum of K and L.

Corollary 1.4 and Corollary 1.5 were established by Wei, Wang and Lu [19].

2. Notations and Background Materials

2.1. L_p Radial Minkowski Combinations

The L_p radial Minkowski combination of star bodies was introduced as follows [27, 7, 28]: For $K,L\in\mathcal{S}_o^n$, real $p\neq 0$ and $\lambda,\mu\ge 0$ (nor both 0), the L_p radial Minkowski combination, $\lambda\cdot K\widetilde{+}_p\mu\cdot L$, of K and L is defined by

$$\rho(\lambda\cdot K\widetilde{+}_p\mu\cdot L,\cdot)^p=\lambda\rho(K,\cdot)^p+\mu\rho(L,\cdot)^p. \tag{6}$$

Here "$\widetilde{+}_p$" denotes the L_p radial Minkowski sum, $\lambda\cdot K$ denotes the L_p radial Minkowski scalar multiplication and $\lambda\cdot K=\lambda^{1/p}K$. The case $p=1$ yields the radial Minkowski combination $\lambda\cdot K\widetilde{+}\mu\cdot L$.

Let $p=-q$ ($q\ge 1$) in (6), then $\lambda\cdot K\widetilde{+}_{-q}\mu\cdot L$ is called the L_q harmonic radial combination of star bodies K and L [1, 7].

In 2015, Wang and Wang [29] defined the L_p radial Blaschke combinations of star bodies as follows: For $K,L\in\mathcal{S}_o^n$, $n>p>0$ and $\lambda,\mu\ge 0$ (not both 0), the L_p radial Blaschke combination, $\lambda\circ K\widehat{+}_p\mu\circ L\in\mathcal{S}_o^n$, of K and L is defined by

$$\rho(\lambda\circ K\widehat{+}_p\mu\circ L,\cdot)^{n-p}=\lambda\rho(K,\cdot)^{n-p}+\mu\rho(L,\cdot)^{n-p}. \tag{7}$$

Here "$\widehat{+}_p$" denotes the L_p radial Blaschke sum, $\lambda\circ K$ denotes the L_p radial Blaschke scalar multiplication and $\lambda\circ K=\lambda^{1/(n-p)}K$. Clearly, (6) and (7) show that $\lambda\circ K\widehat{+}_p\mu\circ L=\lambda\cdot K\widetilde{+}_{n-p}\mu\cdot L$.

2.2. Dual Mixed Volumes

In 1975, Lutwak [30] introduced the dual mixed volumes as follows: For $K_1,\cdots,K_n\in\mathcal{S}_o^n$, the dual mixed volume $\widetilde{V}(K_1,\cdots,K_n)$ is defined by

$$\widetilde{V}(K_1,K_2,\cdots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho_{K_1}(u)\rho_{K_2}(u)\cdots\rho_{K_n}(u)du.$$

For $K,L\in\mathcal{S}_o^n$ and $i=0,1,2,\cdots,n$, we write $\widetilde{V}_i(K,L)$ to denote the dual mixed volume $\widetilde{V}_i(\underbrace{K,\cdots,K}_{n-i},\underbrace{L,\cdots,L}_{i})$. Similarly, we use $\widetilde{W}_i(K,L)=\widetilde{V}_i(\underbrace{K,\cdots,K}_{n-i-1},\underbrace{B,\cdots,B}_{i},L)$ to denote the dual mixed quermassintegrals of K and L. If let i be real, then an extension of the dual mixed volume $\widetilde{V}_i(K,L)$ is that for $K,L\in\mathcal{S}_o^n$ and $i\in\mathbb{R}$:

$$\widetilde{V}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}\rho(L,u)^idu. \tag{8}$$

In the same way, for $K,L\in\mathcal{S}_o^n$ and $i\in\mathbb{R}$, we define the dual mixed quermassintegrals of K and L by

$$\widetilde{W}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i-1}\rho(L,u)du. \tag{9}$$

Let $L=B$ in (8) (or $L=K$ in (9)), then the dual quermassintegrals, $\widetilde{W}_i(K)$, of $K\in\mathcal{S}_o^n$ is given by

$$\begin{aligned}\widetilde{W}_i(K)&=\widetilde{V}_i(K,B)=\widetilde{W}_i(K,K)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)du.\end{aligned}\tag{10}$$

Further let $i=0$ in (10), then we have the following polar coordinate formula for the volume of a body K:

$$V(K)=\widetilde{W}_0(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)du.$$

For the above dual mixed quermassintegrals, corresponding the Minkowski inequality is stated that [10]: *If $K,L\in\mathcal{S}_o^n$ and real i satisfies $0\le i<n-1$, then*

$$\widetilde{W}_i(K,L)\le\widetilde{W}_i(K)^{\frac{n-i-1}{n-i}}\widetilde{W}_i(L)^{\frac{1}{n-i}},\tag{11}$$

with equality if and only if K and L are dilates.

3. Proofs of the Main Results

In this section we will prove Theorems 1.1 and 1.2. To see this, the following lemmas is required.

Lemma 3.1 [30] If $K,L\in\mathcal{S}_0^n$, and reals i,j,k satisfy $i<j<k$, then

$$\widetilde{V}_j(K,L)^{k-i}\le\widetilde{V}_i(K,L)^{k-j}\widetilde{V}_k(K,L)^{j-i},\tag{12}$$

with equality if and only if K and L are dilates.

Lemma 3.2 [10] If $L\in\mathcal{S}_o^n$, $i=0,1,\cdots,n-2$, then

$$\widetilde{W}_{n-1}(\Psi_iL)=r_\Psi\widetilde{W}_{i+1}(L).\tag{13}$$

Proof of Theorem 1.1 In (12), let $L=B$, $j=n-1$ and $k=n$, and notice that $\widetilde{V}_n(K,B)=\omega_n$ by (8), we have for $i<n-1$,

$$\widetilde{W}_{n-1}(K)^{n-i}\le\omega_n^{n-i-1}\widetilde{W}_i(K),\tag{14}$$

with equality if and only if K is a ball centrad at the origin.

Taking Ψ_jK for K in (14), we get for $j=0,1,\cdots,n-1$,

$$\widetilde{W}_{n-1}(\Psi_jK)^{n-i}\le\omega_n^{n-i-1}\widetilde{W}_i(\Psi_jK).$$

This and (13) yield

$$r_\Psi^{n-i}\widetilde{W}_{j+1}(K)^{n-i}\le\omega_n^{n-i-1}\widetilde{W}_i(\Psi_jK),$$

with equality if and only if ψ_jK is a ball centrad at the origin.

Thus, we have

$$\widetilde{W}_i(\Psi_jK)\ge\frac{r_\Psi^{n-i}\widetilde{W}_{j+1}(K)^{n-i}}{\omega_n^{n-i-1}},$$

with equality if and only if ψ_jK is a ball centrad at the origin. This gives Theorem 1.1. $\square$

Now we give the proof of Theorem 1.2. Here, we prove a lemma as follows:

Lemma 3.3 Let $K,L\in\mathcal{S}_o^n$, real $p\neq0$, and j is an integer. If $p<0$ and $0\le j<n-1$, then

$$\widetilde{W}_j(K\widetilde{+}_pL,Q)^{\frac{p}{n-j-1}}\ge\widetilde{W}_i(K,Q)^{\frac{p}{n-j-1}}+\widetilde{W}_i(L,Q)^{\frac{p}{n-j-1}};\tag{15}$$

if $0<p<n-1$ and $0\le j<n-p-1$, then

$$\widetilde{W}_j(K\widetilde{+}_pL,Q)^{\frac{p}{n-j-1}}\le\widetilde{W}_i(K,Q)^{\frac{p}{n-j-1}}+\widetilde{W}_i(L,Q)^{\frac{p}{n-j-1}}.\tag{16}$$

In each inequality, equality holds if and only if K and L are dilates.

Proof. From (6), (9) and the Minkowski integral inequality, if $p<0$ and $0\le j<n-1$, then $\frac{n-j-1}{p}<0$. Thus, for any $Q\in\mathcal{S}_o^n$,

$$\begin{aligned}\widetilde{W}_j(K\widetilde{+}_pL,Q)^{\frac{p}{n-j-1}}&=\left[\frac{1}{n}\int_{S^{n-1}}\rho_{K\widetilde{+}_pL}(u)^{n-j-1}\rho_Q(u)du\right]^{\frac{p}{n-j-1}}=\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho_{K\widetilde{+}_pL}(u)^p\right)^{\frac{n-j-1}{p}}\rho_Q(u)du\right]^{\frac{p}{n-j-1}}\\&=\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho_K(u)^p+\rho_L(u)^p\right)^{\frac{n-j-1}{p}}\rho_Q(u)du\right]^{\frac{p}{n-j-1}}\ge\left[\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-j-1}\rho_Q(u)du\right]^{\frac{p}{n-j-1}}\\&+\left[\frac{1}{n}\int_{S^{n-1}}\rho_L(u)^{n-j-1}\rho_Q(u)du\right]^{\frac{p}{n-j-1}}=\widetilde{W}_j(K,Q)^{\frac{p}{n-j-1}}+\widetilde{W}_j(L,Q)^{\frac{p}{n-j-1}}.\end{aligned}$$

This yields inequality (15).

According to the equality condition of Minkowski integral inequality, we easily see equality holds in (15) if and only if K and L are dilates.

Similarly, if $0<p<n-1$ and $0\le j<n-p-1$, then $\frac{n-j-1}{p}>1$. Hence, by the Minkowski integral inequality, we know that inequality (15) is reverse, i.e., inequality (16) and its equality condition are obtained. $\square$

Lemma 3.4 [10] If $K,L\in\mathcal{S}_o^n$ and $i,j=0,1,\cdots,n-2$, then

$$\widetilde{W}_i(K,\Psi_jL)=\widetilde{W}_j(L,\Psi_iK).\tag{17}$$

Proof of Theorem 1.2 For $i = 0, 1, \cdots, n-2$, if $p < 0$ and $0 \le j < n-1$, then by (15) and (17) we have for any $M \in \mathcal{S}_o^n$,

$$\begin{aligned}\widetilde{W}_i(M, \Psi_j(K \widetilde{+}_p L))^{\frac{p}{n-j-1}} &= \widetilde{W}_j(K \widetilde{+}_p L, \Psi_i M)^{\frac{p}{n-j-1}} \\ &\ge \widetilde{W}_j(K, \Psi_i M)^{\frac{p}{n-j-1}} + \widetilde{W}_j(L, \Psi_i M)^{\frac{p}{n-j-1}} \\ &= \widetilde{W}_i(M, \Psi_j K)^{\frac{p}{n-j-1}} + \widetilde{W}_i(M, \Psi_j L)^{\frac{p}{n-j-1}}.\end{aligned}$$

This together with Minkowski inequality (11), and notice that $\frac{p}{n-j-1} < 0$, we get

$$\begin{aligned}&\widetilde{W}_i(M, \Psi_j(K \widetilde{+}_p L))^{\frac{p}{n-j-1}} \ge \widetilde{W}_i(M, \Psi_j K)^{\frac{p}{n-j-1}} + \widetilde{W}_i(M, \Psi_j L)^{\frac{p}{n-j-1}} \\ &\ge \widetilde{W}_i(M)^{\frac{p(n-i-1)}{(n-i)(n-j-1)}} \left[\widetilde{W}_i(\Psi_j K)^{\frac{p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{p}{(n-i)(n-j-1)}}\right].\end{aligned}$$

Let $M = \Psi_j(K \widetilde{+}_p L)$ in above inequality, we obtain inequality (4).

By the equality conditions of inequalities (11) and (15), we see that equality holds in (4) if and only if K and L are dilates.

In the same method, if $0 < p < n-1$ and $0 \le j < n-p-1$, then inequality (5) and its equality condition can be obtained. □

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