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A Galerkin Finite Element Method for Two-Point Boundary Value Problems of Ordinary Differential Equations

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Abstract: In this paper, we present a new method for solving two-point boundary value problem for certain ordinary differential equation. The two point boundary value problems have great importance in chemical engineering, deflection of beams etc. In this study, Galerkin finite element method is developed for inhomogeneous second-order ordinary differential equations. Several examples are solved to demonstrate the application of the finite element method. It is shown that the finite element method is simple, accurate and well behaved in the presence of singularities.

Keywords: Exact Solution, Two-Point Value Boundary Problem, Finite Element Method

1. Introduction

Two-point boundary-value problems in ordinary differential equations occur in many branches of physics; examples include the two-dimensional, incompressible, one-dimensional heat transfer, boundary layer equations, etc. The corresponding ordinary differential equations can be nonlinear or linear but with complex coefficients. If the differential equation is nonlinear or linear but with complex coefficients, a closed form analytic solution is, in general, difficult to obtain, if not possible. Therefore, a numerical solution is sought. Many researchers have developed numerical technique to study the numerical solution of two point boundary value problems. Villadsen and Stewart [5] proposed solution of boundary value problem by orthogonal collocation method. Jang [6] proposed the solution of two-point boundary value problem by the extended Adomian decomposition method. The Galerkin-finite element method is well known numerical technique for the numerical solution of differential equations. Dogan [7] proposed the Galerkin-finite element approach for the numerical solutions of Burgers' equation. Sengupta et al. [8] carried out Gakerkin finite element methods for wave problems. Kaneko et al. [9] discussed the Discontinuous Galerkin-finite element method for parabolic problems. EI-Gebeily et al. [10] studied the finite element- Galerkin method for singular self-adjoint differential equations. Sharma et al. [11] proposed Galerkin-finite Element Methods for numerical solution of advection-diffusion equation. Onah [12] proved the asymptotic convergence of the solution of a parabolic equation by using two methods namely, the Galerkin method expressed in terms of linear splines and the Finite Element Collocation method expressed by cubic spline basis functions. In this paper, we consider the following inhomogeneous second order differential equation

$$\begin{cases} u''(x)+p(x)u'(x)+q(x)u(x)=f(x), & \alpha<x<\beta \\ u(\alpha)=0 \\ u(\beta)=0 \end{cases} \tag{1.1}$$

where $p(x)=\mathbb{C}^2[\alpha,\beta]$ $p(x)\geq\lambda>0$ in $[\alpha,\beta]$, $q(x)=\mathbb{C}^1[\alpha,\beta]$, $q(x)\geq 0$ on $[\alpha,\beta]$ and $f(x)=\mathbb{C}^1[\alpha,\beta]$

We assume that problem (1.1) has a unique solution $u(x)$

In the present work, we use Galerkin-finite element method for the numerical solution of two point boundary value problems. The approach is simple and effective.

The remaining part of the article is organised as follows. In Section 2, we shall first reformulate (1.1) as a variational problem in the space variables x .We shall then define a Galerkin approximation $u(x)$ to the solution of (1.1) by requiring that u lie in a finite-dimensional space of functions, also an error estimate is given. The Full Discretized system arising from either of the spatial discretisations is given in Section 3. In section 4 of this paper, we shall make some direct applications of approximation theory to some test problems. Finally, Section 5 concludes the article with final remarks

2. Formulation of the Variational Problem and Galerkin Approximations

This problem may also be stated in weak form: find $u \in H_0^1([\alpha,\beta])$ such that

$$\Theta(u,w) = (f,w) \text{ for } w \in H_0^1([\alpha,\beta]) \tag{2.1}$$

where

$$\Theta(u,w) = \int_\alpha^\beta \left(-\frac{dw}{dx}\frac{du}{dx} + wp(x)\frac{du}{dx} + wq(x)u \right) dx \,,\ \langle u,w\rangle = \int_\alpha^\beta uw dx \tag{2.1*}$$

The (*standard*) Galerkin method for approximating the solution u of (2.1) amounts to constructing a family of finite–dimensional subspaces $\{S_h\}$ $0 < h < 1$, and seeking $u_h \in S_h$ satisfying the linear system of equations

$$\Theta(u_h,\chi) = (f,\chi) \text{ for } \chi \in S_h \tag{2.2}$$

We shall assume that the data are such that the unique solution u of (2.1) belongs to $u \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)$ and satisfies the elliptic regularity estimate that for some $C > 0$, independent of f and u we have

$$\|u\|_2 \le C\|f\| \tag{2.3}$$

Under our hypotheses, a unique solution u_h of (2.2) exists and satisfies

$$\|u-u_h\|_1 \le C\inf_{x\in S_h}\|u-\chi\|_1 \tag{2.4}$$

for some constant C independent of *h*. The existence-uniquenss of $u_h \in S_h$ is guaranteed by Lax-Milgram theorem applied to the Hilbert space $\left(S_h, \|\cdot\|\right)$ the proof of Lax-Milgram theorem is given in Appendix A. Assuming that

$$\inf_{x\in S_h}\{\|\varphi-\chi\| + h\|\varphi-\chi\|_1\} \le Ch^2\|\varphi\|_2 \quad \varphi \in \left(H_0^1(\Omega)\cap H^2(\Omega)\right) \tag{2.5}$$

we obtain from (2.4) the optimal–rate $H^1(\Omega)$ –error estimate

$$\|u-u_h\|_1 \le Ch\ \|u\|_2 \tag{2.6}$$

The L^2–error estimate is obtained by the "Nitsche trick", by letting $e = u - u_h$ and considering $w \in \left(H_0^1(\Omega)\cap H^2(\Omega)\right)$ the solution of the problem

$$\Theta(w,\varphi) = (e,\varphi) \text{ for } \varphi \in H_0^1(\Omega) \tag{2.7}$$

Then $\|e\|^2 = (e,e) = \Theta(w,e) = \Theta(e,w) = \Theta(e,w-\chi)$ for any $\chi \in S_h$,

By the continuity of L in $H^1(\Omega) * H^1(\Omega)$ we have then

$$\|e\|^2 = C\|e\|_1\|w-\chi\|_1 \le C\|e\|_1 h\|w\|_2 \le Ch\|e\|_1\|e\|. \tag{2.8}$$

Hence $\|e\| \le Ch\|e\|_1 \le Ch^2\|u\|_2$

In general, assuming that for some integer $r \ge 2$

$$\inf_{x\in S_h}\{\|w-\chi\| + h\|w-\chi\|_1\} \le Ch^r\|w\|_r \text{ for } w \in \left(H_0^1(\Omega)\cap H^r(\Omega)\right) \tag{2.9}$$

where $\|.\|$ denotes the norm in $H^s(\Omega)$, $\|.\| = \|.\|_0$ and the Nitsche argument give $\|e(t)\| + h\|e(t)\|_1 \le Ch^r\|u\|_r$ $u \in \left(H_0^1(\Omega)\cap H^r(\Omega)\right)$

3. Fully Discretized Finite Element Models

We shall approximate the solution of (2.2) by requiring that u and χ lie in $\{S_h\}$ $0 < h < 1$. Let $\chi_\kappa \in S_h$ for $\kappa = 1,2,...,N$. Assume that the set $\chi_1,...,\chi_N$ is linearly independent. Denote by Υ the subspace spanned by $\chi_1,...,\chi_N$,let $\{\chi_\kappa\}_{\kappa=1}^N$ be a basis of S_h where $N = \dim S_h$ .We shall approximate u of (2.1) by a function

$$u_h(x) = \sum_{\kappa=1}^N c_\kappa \chi_\kappa(x) \tag{3.1}$$

$$\Theta(u_h,\chi) = (f,\chi) \text{ for } \chi \in H_0^1(\Omega) \ \ \chi \in S_h \tag{3.2}$$

Substituting this expression for u_h in (2.2) and taking $\chi = \chi_\kappa, \kappa = 1,...,N$ we see that

$$Gc = f_h \tag{3.3}$$

Where G is the NxN matrix defined by $G = G_{\kappa j} = \Theta(\chi_\kappa,\chi_j) = \int_\alpha^\beta \left(\chi_\kappa'\chi' + p(x)\chi_\kappa\chi_j' + q(x)\chi_\kappa\chi_j\right)dx$, $1 \le \kappa, j \le N$ $C = [c_1,\dots,c_N]^T$ and $f_h = [(f,\chi_1),\dots,(f,\chi_N)]^T$. Where G is a positive definite matrice.

4. Numerical Example

In this section, some numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using MatlabR2012b. The versatility and accuracy of proposed method is measured using L_∞.

$$L_\infty = \|u-U_N\|_\infty = \max_j \left|u_j - (U_N)_j\right|$$

Example 1. Considering equation

$$u''(x)+p(x)u'(x)+q(x)u(x)=f(x), \quad \alpha=0\le x\le \beta=1 \qquad (4.1)$$

with boundary conditions

$$u(0)=0$$
$$u(1)=0$$

where the function $p(x)$ and $q(x)$ are assumed constant, $-3, 2$ respectively, while the function $f(x)$ is assumed 1.

The true solution of this problem is $u(x)=c_1e^{2x}+c_2e^{x}+\frac{1}{2}$,

where $c_1=\frac{1/2}{\exp(1)}, c_2=-\frac{1}{2}\left(1+\frac{1}{\exp(1)}\right)$

Table 1. *$\|\cdot\|_\infty$ concentration errors. Linear elements*

Elements	$\|L\|_\infty$
10	4.5593E-004
40	3.0706E-005
100	4.9871E-006

Example 2. Let's consider the same example with mixed boundary conditions as below

$$u(0)=0$$
$$u'(1)=1$$

The true solution of this problem is $u(x)=c_1e^{2x}+c_2e^{x}+\frac{1}{2}$,

where $c_1=\frac{\left(1+\frac{1}{2}\exp(1)\right)}{(2\exp(2)-\exp(1))}, c_2=\frac{-(1+\exp(2))}{(2\exp(2)-\exp(1))}$

Table 2. *$\|\cdot\|_\infty$ concentration errors. Linear elements*

Elements	$\|L\|_\infty$
10	3.7646E-004
40	2.2106E-005
100	3.5089E-006

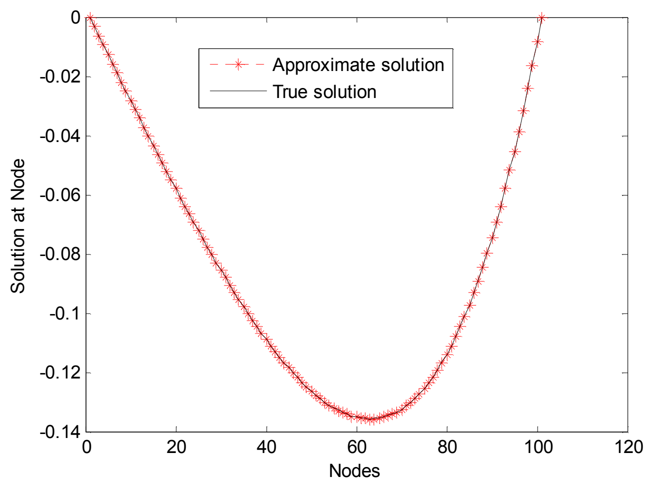

Fig. 1. *Comparison of numerical and exact solution of Example 1. Linear elements*

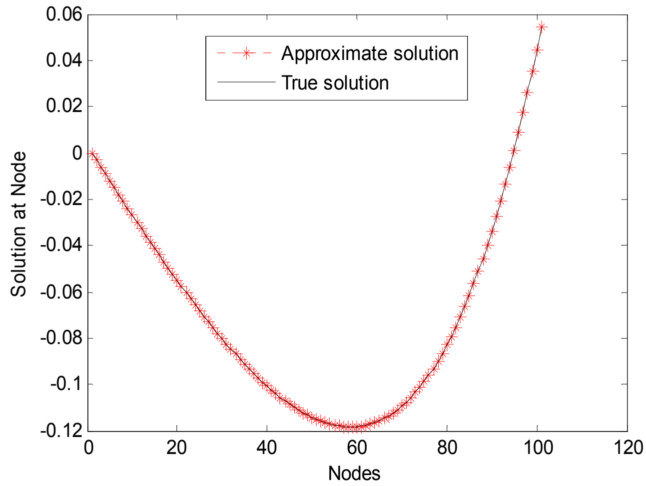

Fig. 2. *Comparison of numerical and exact solution of Example 2. Linear elements*

Example 3. Considering equation

$$x^2u''(x)-2xu'(x)+4u(x)=x^2 \quad 10\le x\le 20$$

with boundary conditions

$$u(10)=0$$
$$u(20)=100$$

The true solution of this problem is $0.00102x^4-0.16667x^2+64.5187\frac{1}{x}$.

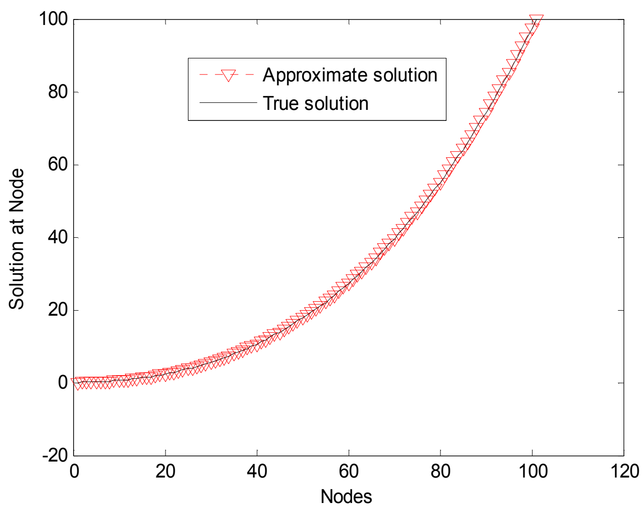

Fig. 3. *Comparison of numerical and exact solution of Example 3. Linear elements*

Table 3. *$\|\cdot\|_\infty$ concentration errors. Linear elements*

Elements	$\|L\|_\infty$
20	6.8E-2
40	5.02E-2
100	1.02E-2

5. Concluding Remarks

In this article, Galerkin-finite element method is proposed to find the approximate solutions of two point boundary value problems. In the solution procedure, the first step is to make weak formulation and then develop finite element formulation. Lastly, weighted average is used for fully discretization. As test problem, three different solutions of three point boundary value problems are chosen. Also, a comparison of numerical and analytical solutions is made and found that the proposed scheme has good accuracy.

Appendix A

Theorem 1 (Lax–Milgram Theorem). Let H be a (real) Hilbert space and let $\Theta(\cdot,\cdot): H\times H \to \mathbb{R}$ be a bilinear form on H which satisfies:

1. $|\Theta(\phi,\psi)| \le c_1 \|\phi\|\|\psi\| \quad \forall \phi,\psi \in H$
2. $\Theta(\phi,\phi) \ge c_2 \|\phi\|^2 \quad \forall \phi \in H$

where c_1, c_2 are positive constants independent of $\phi,\psi \in H$.

Let $F: H \to \mathbb{R}$ be a (real valued) linear functional on H such that.

3. $\exists c_3 > 0 \ \forall \psi \in H \ |F(\psi)| < c_3 |\psi|$

Then there exists a unique $u \in H$ satisfying

$$\Theta(u,w) = F(w) \quad \forall w \in H$$

Moreover,

$$\|u\| \le \frac{1}{c_2}\|F\|$$

Proof. Let $\phi \in H$ be fixed. Then $\Phi: H \to \mathbb{R}$ defined for every $w \in H$ by $\Phi(w) = \Theta(\phi,w)$ defines a continuous linear functional on H. For boundedness observe that for each $w \in H$

$$|\Phi(w)| = |\Theta(\phi,w)| \le c_1 \|\phi\|\|\psi\|$$

Hence $\|\Phi\| \le c_1 \|\phi\| < \infty$

By the Riesz Representation Theorem therefore, there exists a unique element $\hat{\phi}$ such that

$$\Phi(w) = \Theta(\phi,w) = (w,\hat{\phi}) \ \forall w \in H \tag{A.1}$$

Hence for every $\phi \in H$ we define a $\hat{\phi} \in H$ by (A.1) and denote the correspondence $\varphi \mapsto \hat{\phi}$ by $\hat{\phi} = \Lambda\phi$

$$\Theta(\phi,w) = (w,\Lambda\phi) \ \forall w \in H \ \forall \phi \in H \tag{A.2}$$

Now Λ is a linear operator defined on H. We claim now that Λ, defined by (A.2) has a range $\mathrm{Ran}(\Lambda)$ which is a closed subspace of H. Let $\hat{\phi}_n = \Lambda\phi_n$ be a convergent sequence, such that $\hat{\phi}_n \mapsto \hat{\phi}$. Now, since $\Theta(\phi_n,w) = (w,\Lambda\phi_n)$ $\forall w \in H \Rightarrow \Theta(\phi_n - \phi_m, w) = (\Lambda\phi_n - \Lambda\phi_m, w)$ $\forall w \in H$. Choose $w = \phi_n - \phi_m$ and using (2) get $\|\phi_n - \phi_m\| \le \frac{1}{c_2}\|\Lambda\phi_n - \Lambda\phi_m\|$. Hence $\{\phi_n\}$ is a Cauchy sequence in H, there exist $\phi \in H$ such that $\phi_n \mapsto \phi$. We now show that $\hat{\phi} = \Lambda\phi$ thus showing that $\hat{\phi} \in \mathrm{Ran}(\Lambda)$, that $\mathrm{Ran}(\Lambda)$ is closed.

Now $|\Theta(\phi_n,w) - \Theta(\phi,w)| \le C_1 \|\phi_n - \phi\|\|w\| \quad \forall w \in H$ gives that

$$\lim_{n\to\infty} \Theta(\phi_n,w) = \Theta(\phi,w) \ \forall w \in H$$

Also $\Lambda(\phi_n,w) = (\hat{\phi}_n,w) \to (\hat{\phi},w)$ since $|(\hat{\phi}_n,w) - (\hat{\phi},w)| \le \|\hat{\phi}_n - \hat{\phi}\|\|w\|$. Since $\Theta(\phi_n,w) = \Lambda(\phi_n,w)$ $\forall w \in H \Rightarrow \Theta(\phi,w) = (\hat{\phi},w)$. Hence $\mathrm{Ran}(\Lambda)$ is closed. Also we claim that $\mathrm{Ran}(\Lambda) = H$

Given F on H, by Riesz representation $\exists! \xi \in H$ such that $F(w) = (\xi,w)$ $\forall v \in H$. Since $\mathrm{Ran}(\Lambda) = H$ $\exists u \in H$ such that $\Lambda u = \xi$. Hence $\exists u$ such that

$$F(w) = (\Lambda u, w) = \Theta(u,w) \ \forall w \in H$$

For uniqueness, suppose that $\exists u_1 \ne u_2$ such that $\Theta(u_1,w) = F(w) = \Theta(u_2,w)$ $\forall w \in H$. Hence $\Theta(u_1 - u_2, w) = 0$ $\forall w \in H \Rightarrow \Theta(u_1 - u_2, u_1 - u_2) \ge c_2 \|u_1 - u_2\|^2 \Rightarrow u_1 = u_2$

Since $\Theta(u,u) = F(u)$, (1), (2) give that $u \ne 0$ $c_2\|u\| \le |F(u)|$ from which $\|u\| \le \frac{1}{c_2}\frac{|F(u)|}{\|u\|}$. Hence

$$\|u\| \le \sup_{w\ne 0} \frac{1}{c_2}\frac{|F(w)|}{\|w\|} = \frac{1}{c_2}\|F\|$$

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